# CS 343: Artificial Intelligence

## Hidden Markov Models

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[These slides based on those of Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley.  All CS188 materials are available at http://ai.berkeley.edu.]

# Announcements

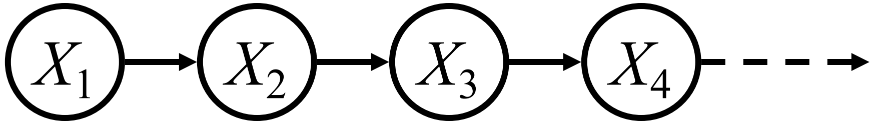
- Homework 4: Probability, Bayes Net
  - Due Monday 3/27 at 11:59 pm
- Programming 4: Bayes Nets
  - Due Wednesday 3/19 at 11:59 pm
- Homework 5: HMMs, Particle Filtering, Naive Bayes, ML Concepts
  - Due Monday 4/10 at 11:59 pm
  - Start early!
- Final Project: Capture the Flag Contest
  - Optional but with extra credits
  - Qualification Due: Wednesday 4/12, 11:59 pm
  - Tournament Due: Monday 4/17, 11:59 pm

# Reasoning over Time or Space

- Often, we want to reason about a sequence of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
- Need to introduce time (or space) into our models

# Markov Models

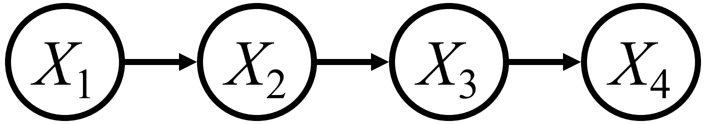
- Value of X at a given time is called the state




$$P(X_1) \qquad P(X_t|X_{t-1})$$

- Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action

# Joint Distribution of a Markov Model




$P(X_1)$ $\qquad$ $P(X_t|X_{t-1})$

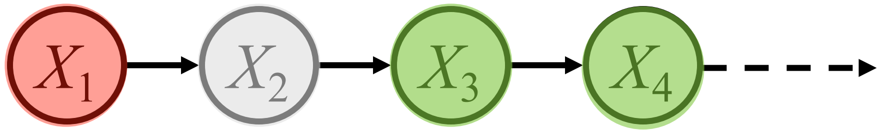
- Joint distribution:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

- More generally:

$$\begin{aligned} P(X_1, X_2, \ldots, X_T) &= P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1}) \\ &= P(X_1)\prod_{t=2}^{T} P(X_t|X_{t-1}) \end{aligned}$$

# Implied Conditional Independencies




- We assumed: $X_3 \perp\!\!\!\perp X_1 \mid X_2$ and $X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$
- Do we also have $X_1 \perp\!\!\!\perp X_3, X_4 \mid X_2$ ?
  - Yes! D-Separation
  - Or, Proof:

$$\begin{aligned} P(X_1 \mid X_2, X_3, X_4) &= \frac{P(X_1, X_2, X_3, X_4)}{P(X_2, X_3, X_4)} \\ &= \frac{P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_2)P(X_4 \mid X_3)}{\sum_{x_1} P(x_1)P(X_2 \mid x_1)P(X_3 \mid X_2)P(X_4 \mid X_3)} \\ &= \frac{P(X_1, X_2)}{P(X_2)} \\ &= P(X_1 \mid X_2) \end{aligned}$$

# Markov Models Recap

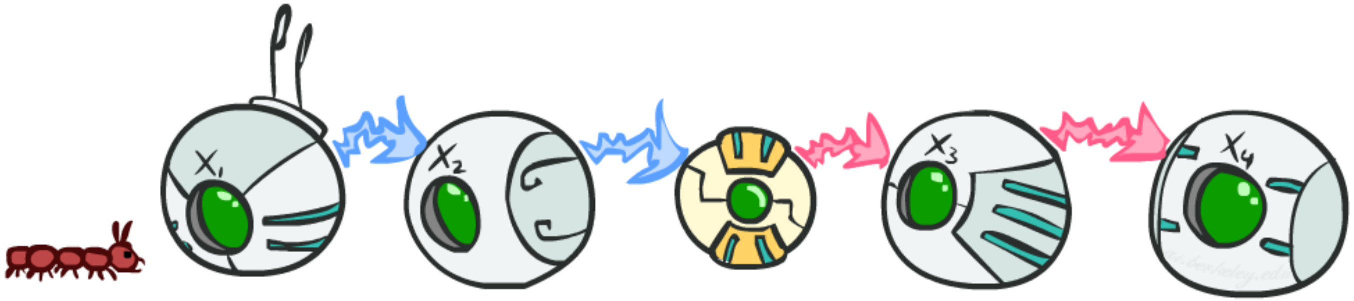
- Explicit assumption for all *t* : $X_t \perp\!\!\!\perp X_1, \ldots, X_{t-2} \mid X_{t-1}$
- Consequence, joint distribution can be written as:

$$P(X_1, X_2, \ldots, X_T) = P(X_1)P(X_2|X_1)P(X_3|X_2)\ldots P(X_T|X_{T-1})$$
$$= P(X_1)\prod_{t=2}^{T} P(X_t|X_{t-1})$$

- Implied conditional independencies:
  - Past variables independent of future variables given the present

  i.e., if $t_1 < t_2 < t_3$ or $t_1 > t_2 > t_3$ then: $X_{t_1} \perp\!\!\!\perp X_{t_3} \mid X_{t_2}$
- Additional explicit assumption: $P(X_t \mid X_{t-1})$ is the same for all *t*

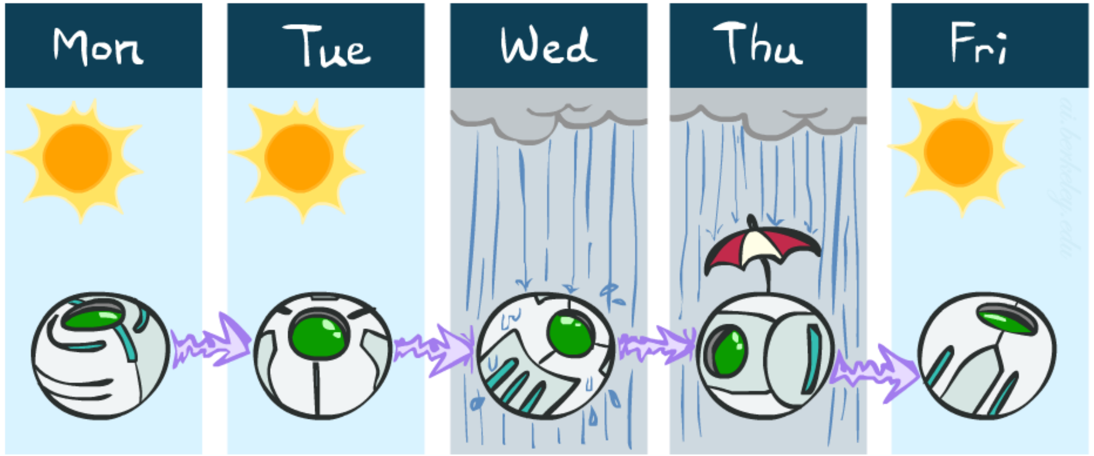
# Conditional Independence




- Basic conditional independence:
  - Past and future independent given the present
  - Each time step only depends on the previous
  - This is the (first order) Markov property (remember MDPs?)

- Note that the chain is just a (growable) BN
  - We can always use generic BN reasoning on it if we truncate the chain at a fixed length

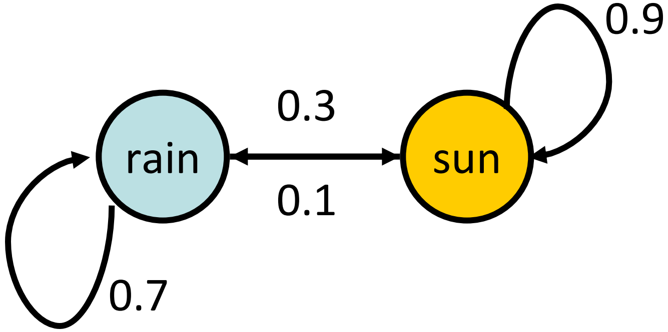
# Example Markov Chain: Weather

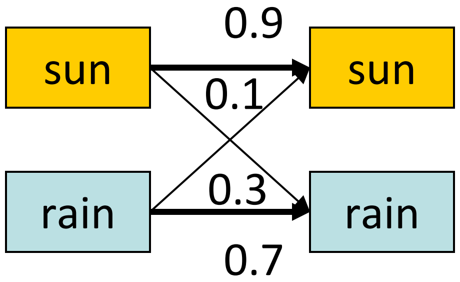



- States: X = {rain, sun}
- Initial distribution: 1.0 sun
- CPT $P(X_t \mid X_{t-1})$:

| $X_{t-1}$ | $X_t$ | $P(X_t \mid X_{t-1})$ |
|---|---|---|
| sun | sun | 0.9 |
| sun | rain | 0.1 |
| rain | sun | 0.3 |
| rain | rain | 0.7 |

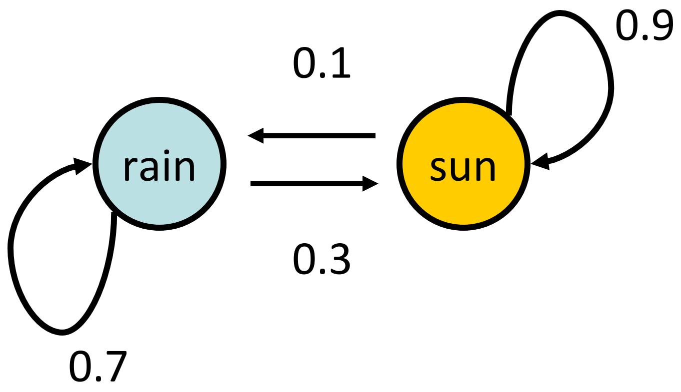
Two new ways of representing the same CPT



# Example Markov Chain: Weather

- Initial distribution: 0.6 sun / 0.4 rain



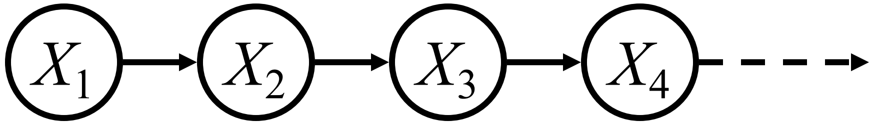

- What is the probability distribution after one step?

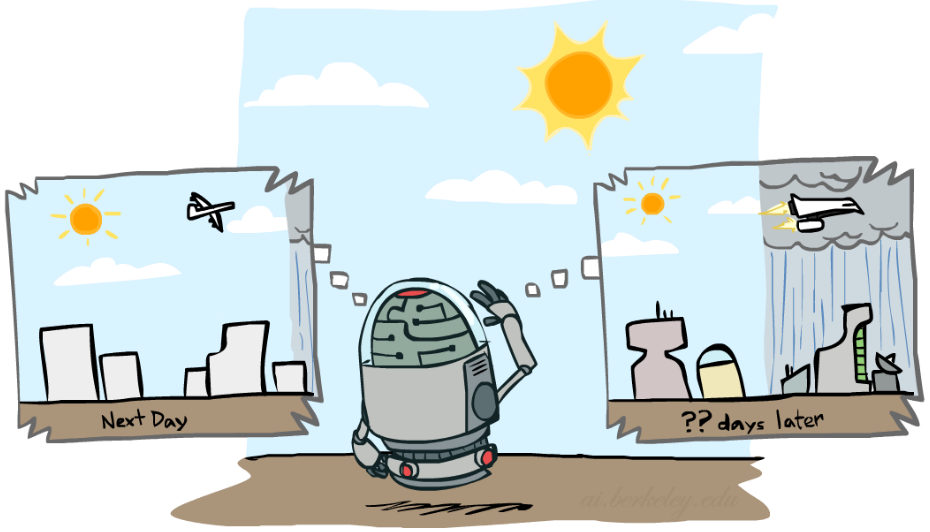
$$P(X_2 = \text{sun}) = P(X_2 = \text{sun}|X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun}|X_1 = \text{rain})P(X_1 = \text{rain})$$

$$= 0.9 * 0.6 + 0.3 * 0.4 \quad = \quad 0.66$$

# Mini-Forward Algorithm

- Question: What's P(X) on some day t?




$$P(x_1) = \text{known}$$

$$P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t)$$

$$= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1})$$

*Recursion*

*Forward simulation*

# Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\underbrace{\begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix}}_{P(X_1)} \quad \underbrace{\begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}}_{P(X_2)} \quad \underbrace{\begin{pmatrix} 0.84 \\ 0.16 \end{pmatrix}}_{P(X_3)} \quad \underbrace{\begin{pmatrix} 0.804 \\ 0.196 \end{pmatrix}}_{P(X_4)} \Rightarrow \underbrace{\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}}_{P(X_\infty)}$$

- From initial observation of rain

$$\underbrace{\begin{pmatrix} 0.0 \\ 1.0 \end{pmatrix}}_{P(X_1)} \quad \underbrace{\begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}}_{P(X_2)} \quad \underbrace{\begin{pmatrix} 0.48 \\ 0.52 \end{pmatrix}}_{P(X_3)} \quad \underbrace{\begin{pmatrix} 0.588 \\ 0.412 \end{pmatrix}}_{P(X_4)} \Rightarrow \underbrace{\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}}_{P(X_\infty)}$$

- From yet another initial distribution $P(X_1)$:

$$\underbrace{\begin{pmatrix} p \\ 1-p \end{pmatrix}}_{P(X_1)} \quad \cdots \Rightarrow \underbrace{\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}}_{P(X_\infty)}$$

# Stationary Distributions

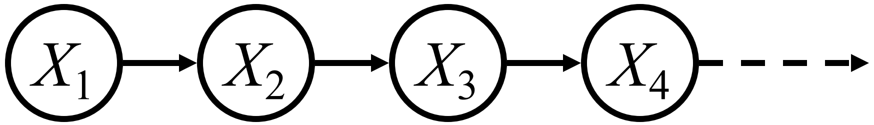
- For most chains:
  - Influence of the initial distribution gets less and less over time.
  - The distribution we end up in is independent of the initial distribution

- Stationary distribution:
  - The distribution we end up with is called the stationary distribution $P_\infty$ of the chain
  - It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$

# Example: Stationary Distributions

- Question: What's P(X) at time t = infinity?




Remember:

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$

Also: $P_\infty(sun) + P_\infty(rain) = 1$

$$P_\infty(sun) = P(sun|sun)P_\infty(sun) + P(sun|rain)P_\infty(rain)$$
$$P_\infty(rain) = P(rain|sun)P_\infty(sun) + P(rain|rain)P_\infty(rain)$$

$$P_\infty(sun) = 0.9P_\infty(sun) + 0.3P_\infty(rain)$$
$$P_\infty(rain) = 0.1P_\infty(sun) + 0.7P_\infty(rain)$$

$$P_\infty(sun) = 3P_\infty(rain)$$
$$P_\infty(rain) = 1/3P_\infty(sun)$$

Also: $P_\infty(sun) + P_\infty(rain) = 1$

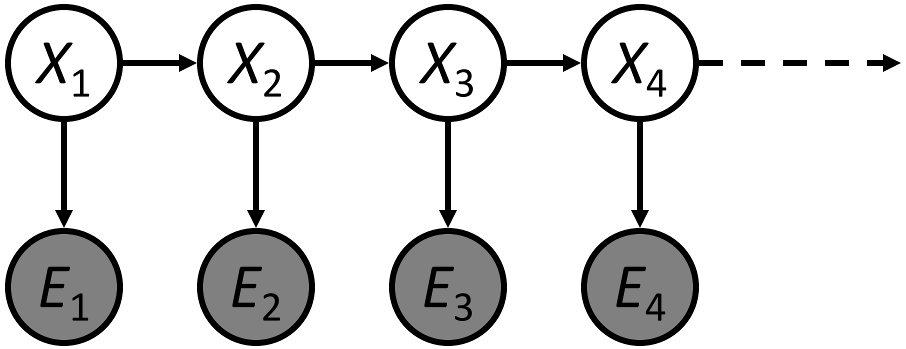
$$P_\infty(sun) = 3/4$$
$$P_\infty(rain) = 1/4$$

| $X_{t-1}$ | $X_t$ | $P(X_t \mid X_{t-1})$ |
|---|---|---|
| sun | sun | 0.9 |
| sun | rain | 0.1 |
| rain | sun | 0.3 |
| rain | rain | 0.7 |

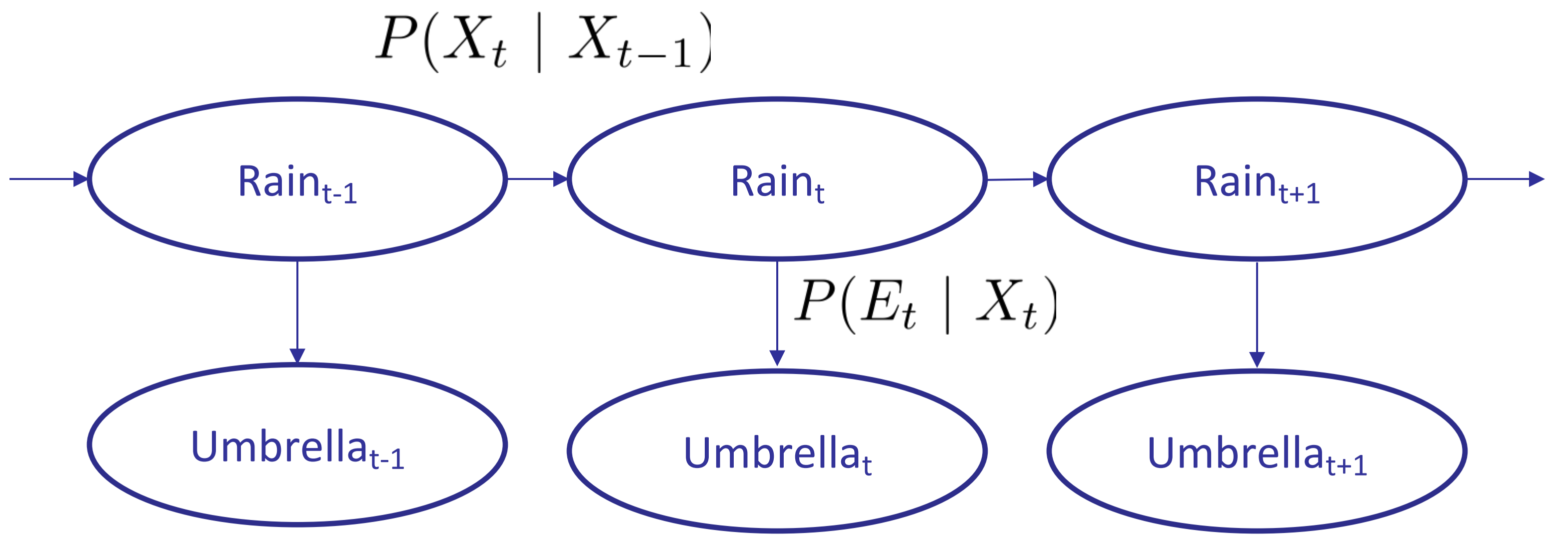
# Hidden Markov Models

# Hidden Markov Models

- Markov chains not so useful for most agents
  - Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states X
  - You observe outputs (effects) at each time step




# Example: Weather HMM

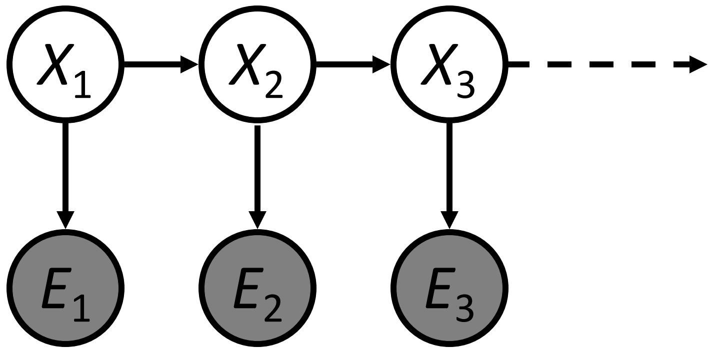



- An HMM is defined by:
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X_t \mid X_{t-1})$
  - Emissions: $P(E_t \mid X_t)$

| $R_t$ | $R_{t+1}$ | $P(R_{t+1} \mid R_t)$ |
|---|---|---|
| +r | +r | 0.7 |
| +r | -r | 0.3 |
| -r | +r | 0.3 |
| -r | -r | 0.7 |

| $R_t$ | $U_t$ | $P(U_t \mid R_t)$ |
|---|---|---|
| +r | +u | 0.9 |
| +r | -u | 0.1 |
| -r | +u | 0.2 |
| -r | -u | 0.8 |

# Joint Distribution of an HMM



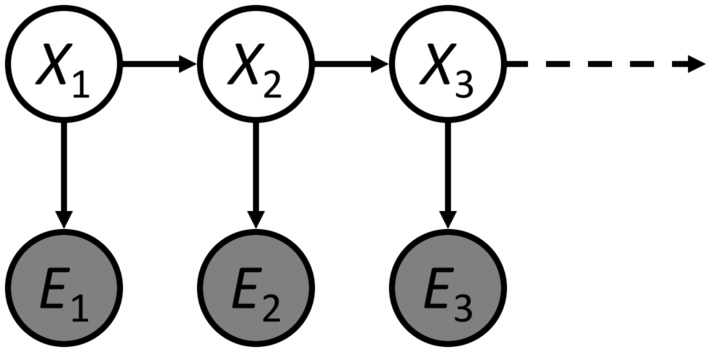

- Joint distribution:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$

- More generally:

$$P(X_1, E_1, \ldots, X_T, E_T) = P(X_1)P(E_1|X_1)\prod_{t=2}^{T} P(X_t|X_{t-1})P(E_t|X_t)$$

# Implied Conditional Independencies




- Many implied conditional independencies, e.g.,

$$E_1 \perp\!\!\!\perp X_2, E_2, X_3, E_3 \mid X_1$$

- To prove them
  - Approach 1: follow similar (algebraic) approach to what we did for Markov models
  - Approach 2: D-Separation

# Real HMM Examples

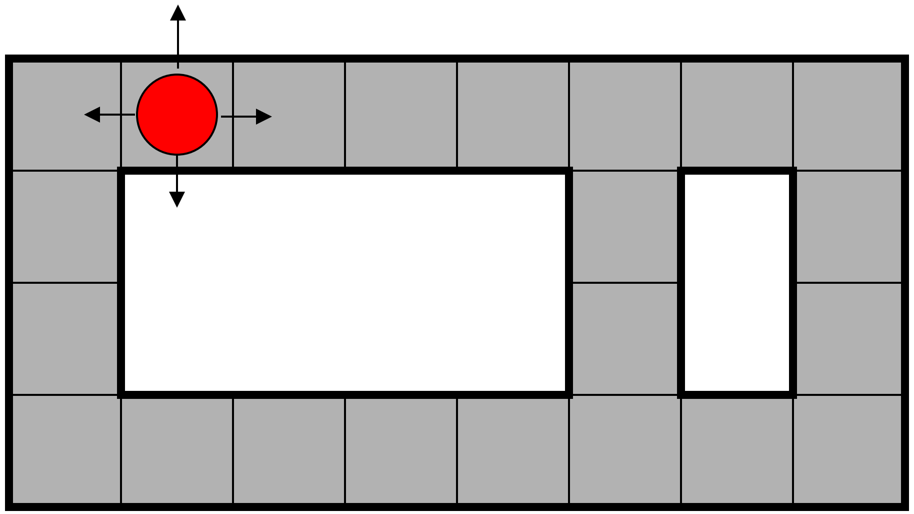
- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options
- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)

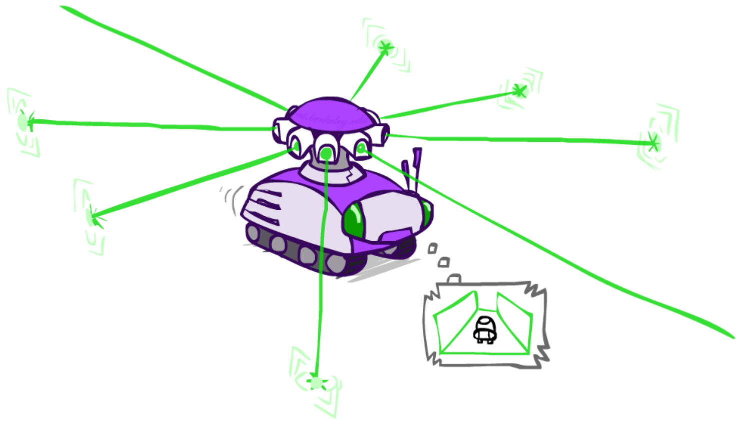
# Filtering / Monitoring

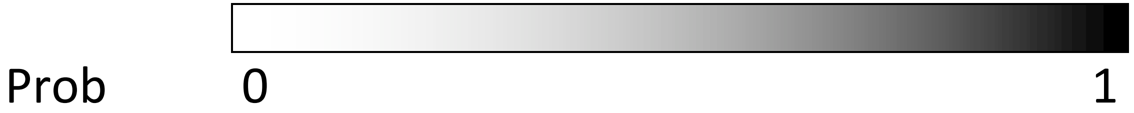
- Filtering, or monitoring, is the task of tracking the distribution $B_t(X) = P_t(X_t \mid e_1, \ldots, e_t)$ (the belief state) over time
- We start with $B_1(X)$ in an initial setting, usually uniform
- As time passes, or we get observations, we update B(X)
- The Kalman filter was invented in the 60's and first implemented as a method of trajectory estimation for the Apollo program

# Example: Robot Localization

*Example from*
*Michael Pfeiffer*





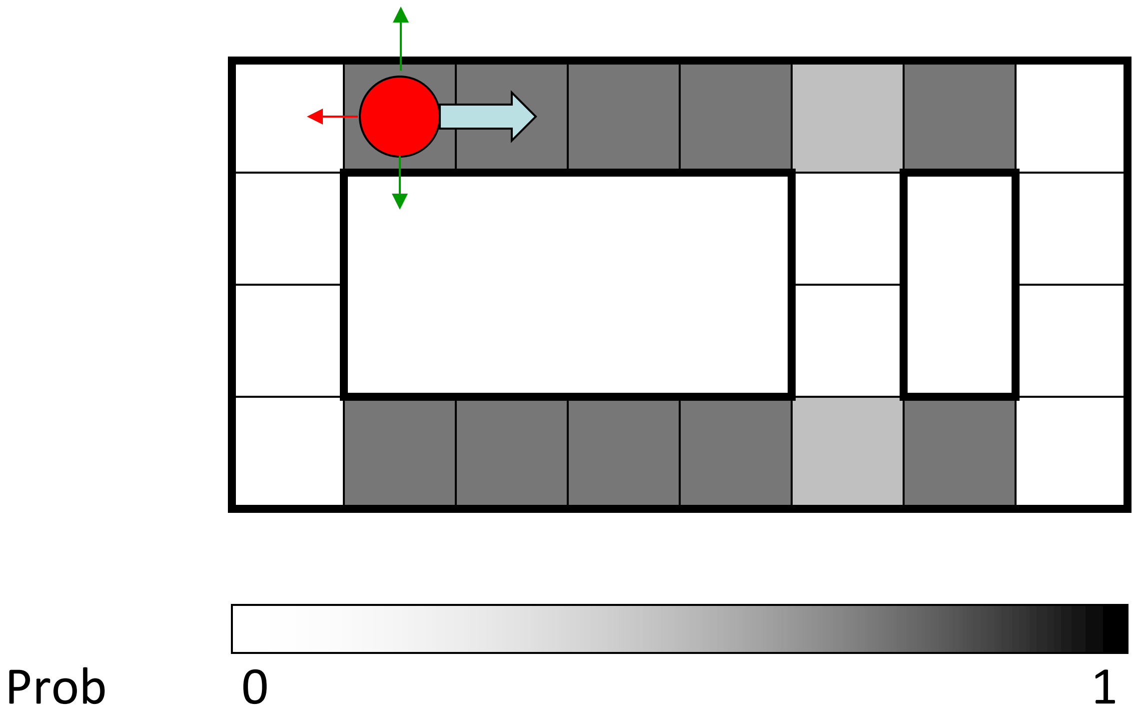


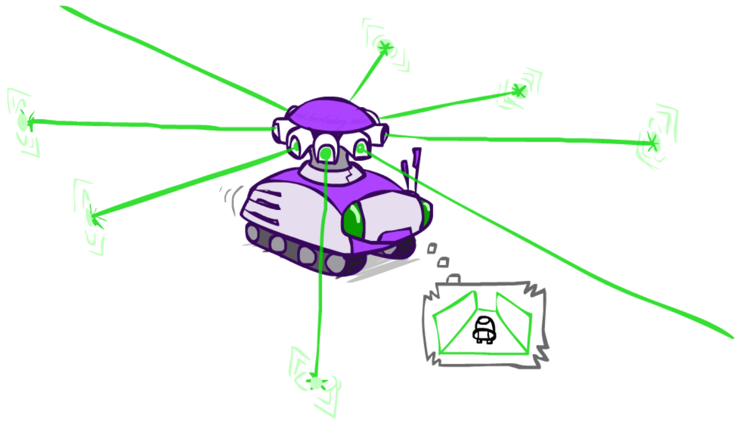

t=0

Sensor model: can read in which directions there is a wall, never more than 1 mistake

Motion model: may not execute action with small prob.

# Example: Robot Localization

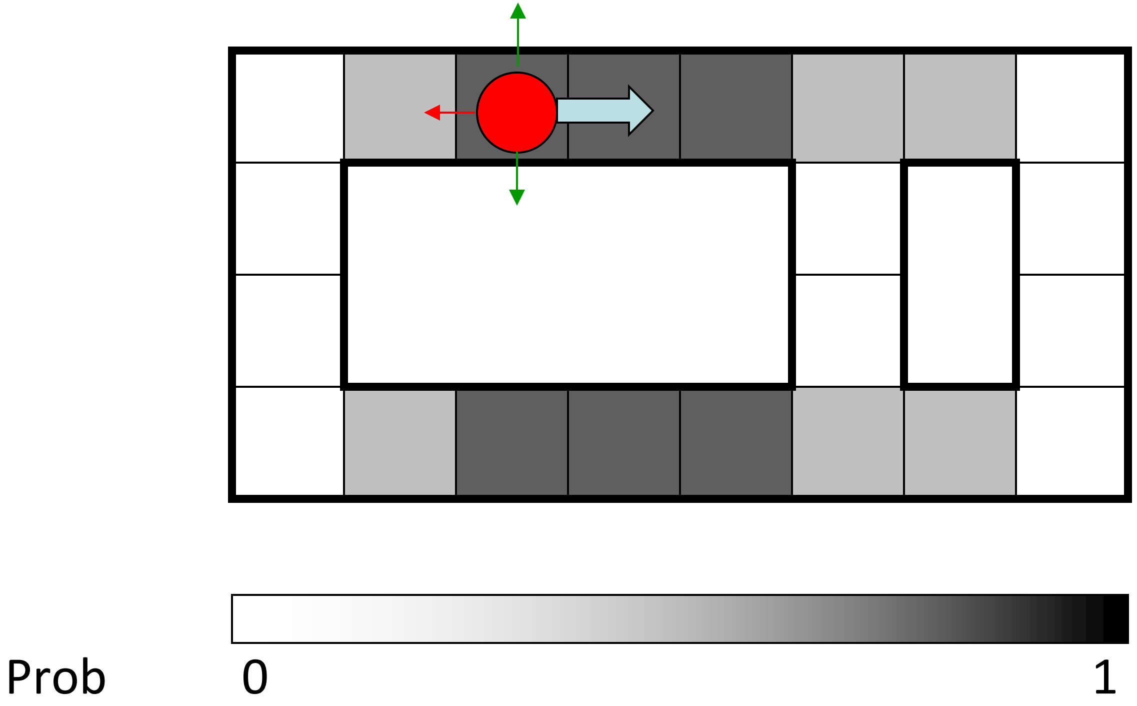


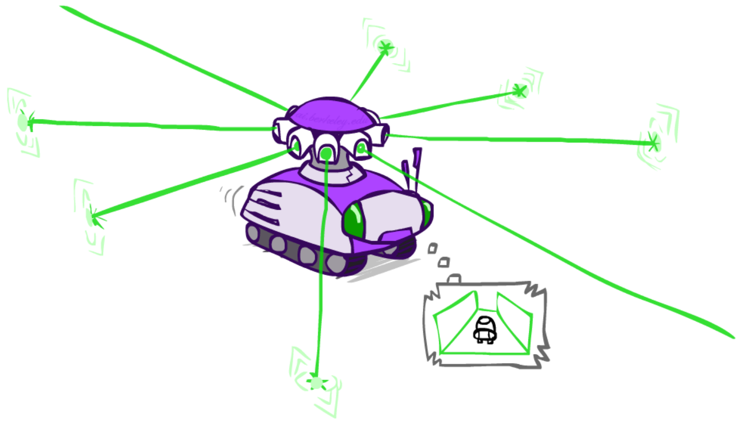



t=1

Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake

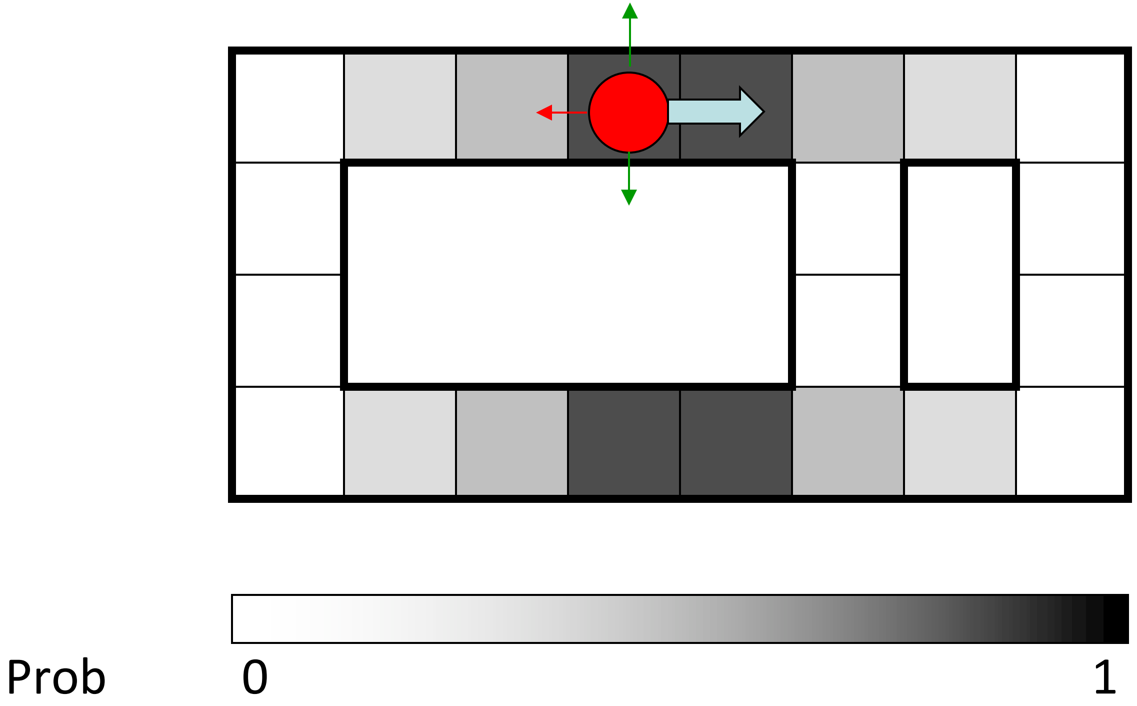
# Example: Robot Localization

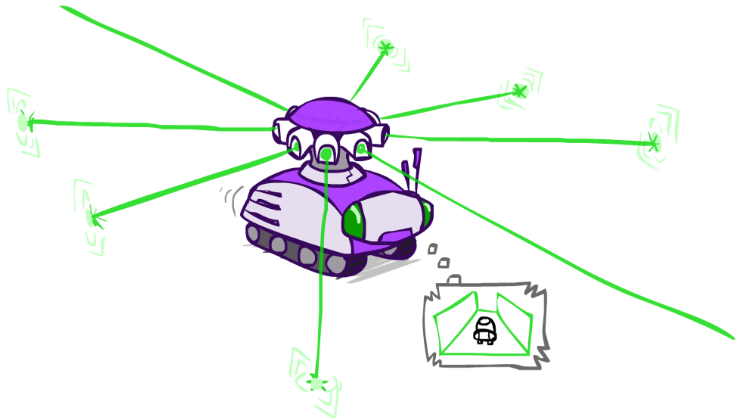


t=2

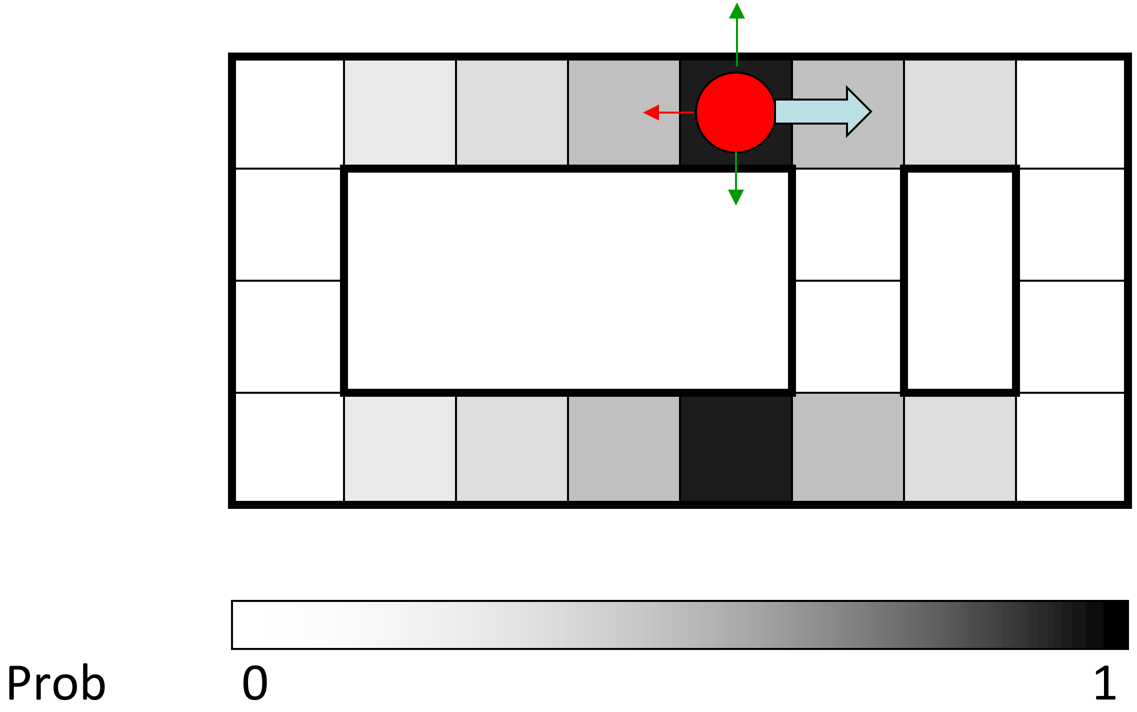
# Example: Robot Localization

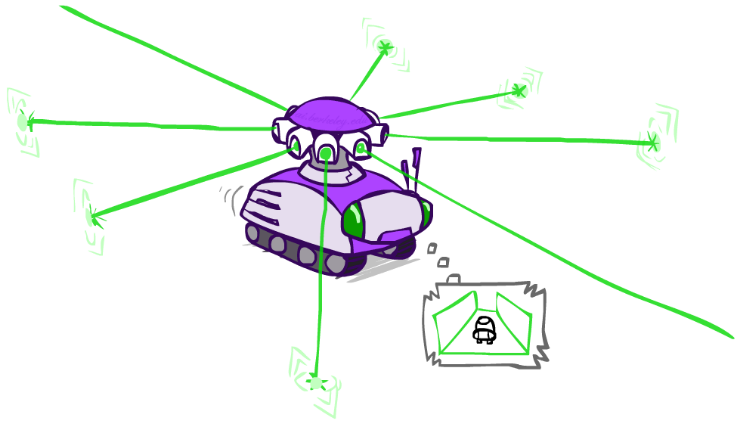


t=3

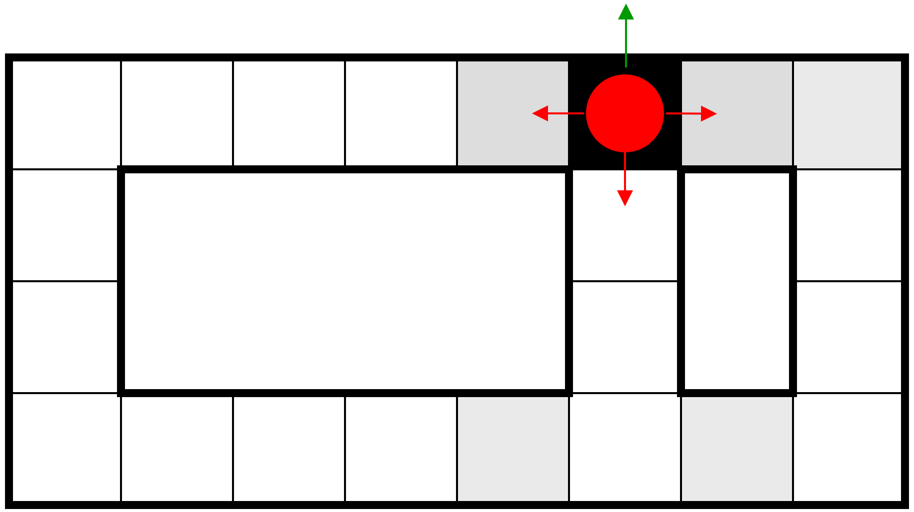
# Example: Robot Localization

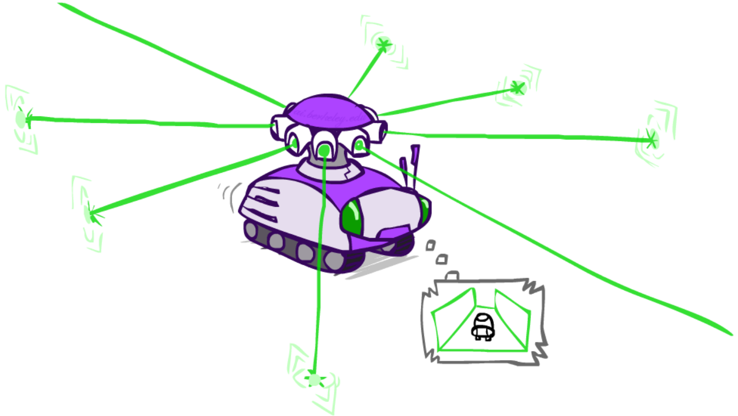


t=4

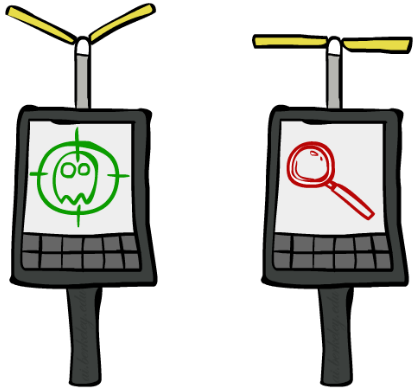
# Example: Robot Localization

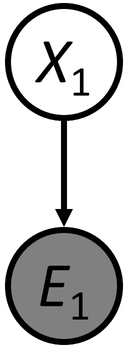


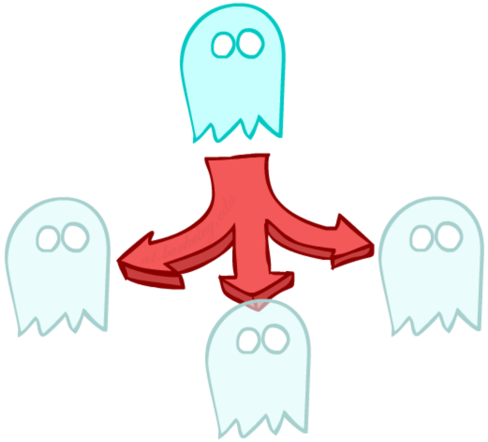



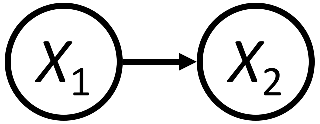
t=5

# Inference: Base Cases








$$P(X_1|e_1)$$

$$P(x_1|e_1) = P(x_1, e_1)/P(e_1)$$

$$\propto_{X_1} P(x_1, e_1)$$

$$= P(x_1)P(e_1|x_1)$$

$$P(X_2)$$

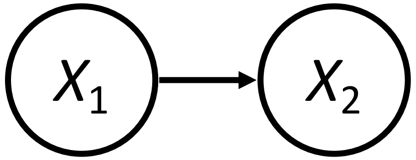
$$P(x_2) = \sum_{x_1} P(x_1, x_2)$$

$$= \sum_{x_1} P(x_1)P(x_2|x_1)$$

# Passage of Time

- Assume we have current belief P(X | evidence to date)

$$B(X_t) = P(X_t|e_{1:t})$$




- Then, after one time step passes:

$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t|e_{1:t})$$

$$= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t})P(x_t|e_{1:t})$$

$$= \sum_{x_t} P(X_{t+1}|x_t)P(x_t|e_{1:t})$$

- Or compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$

- Basic idea: beliefs get "pushed" through the transitions
  - With the "B" notation, we have to be careful about what time step t the belief is about, and what evidence it includes

# Example: Passage of Time

- As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)

| | | | | | |
|---|---|---|---|---|---|
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | 1.00 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |

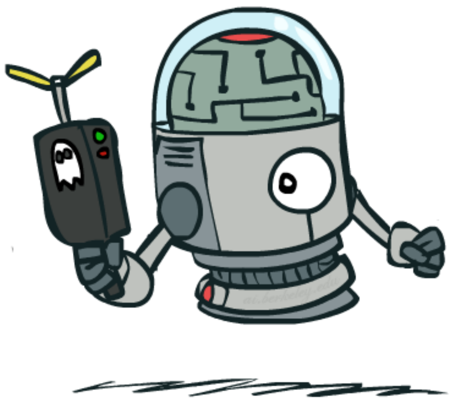
T = 1

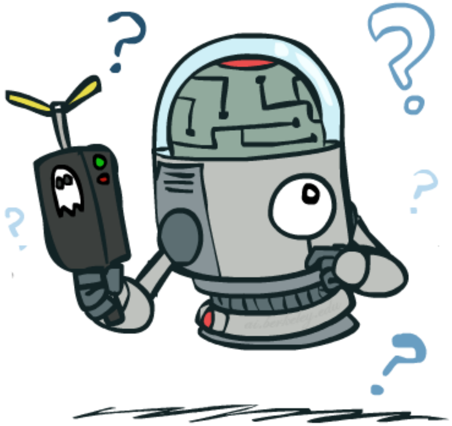
| | | | | | |
|---|---|---|---|---|---|
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | 0.06 | <0.01 | <0.01 | <0.01 |
| <0.01 | 0.76 | 0.06 | 0.06 | <0.01 | <0.01 |
| <0.01 | <0.01 | 0.06 | <0.01 | <0.01 | <0.01 |

T = 2

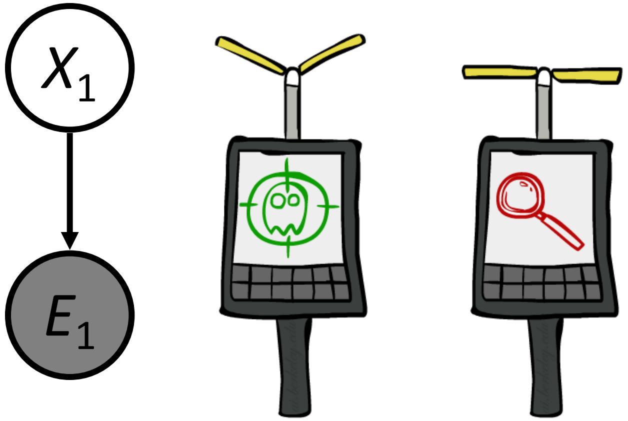
| | | | | | |
|---|---|---|---|---|---|
| 0.05 | 0.01 | 0.05 | <0.01 | <0.01 | <0.01 |
| 0.02 | 0.14 | 0.11 | 0.35 | <0.01 | <0.01 |
| 0.07 | 0.03 | 0.05 | <0.01 | 0.03 | <0.01 |
| 0.03 | 0.03 | <0.01 | <0.01 | <0.01 | <0.01 |

T = 5





# Observation




- Assume we have current belief P(X | previous evidence):

$$B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$$

- Then, after evidence comes in:

$$P(X_{t+1}|e_{1:t+1}) = P(X_{t+1}, e_{t+1}|e_{1:t})/P(e_{t+1}|e_{1:t})$$
$$\propto_{X_{t+1}} P(X_{t+1}, e_{t+1}|e_{1:t})$$
$$= P(e_{t+1}|e_{1:t}, X_{t+1})P(X_{t+1}|e_{1:t})$$
$$= P(e_{t+1}|X_{t+1})P(X_{t+1}|e_{1:t})$$

- Or, compactly:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

- Basic idea: beliefs "reweighted" by likelihood of evidence
- Unlike passage of time, we have to renormalize

# Example: Observation

- As we get observations, beliefs get reweighted, uncertainty "decreases"

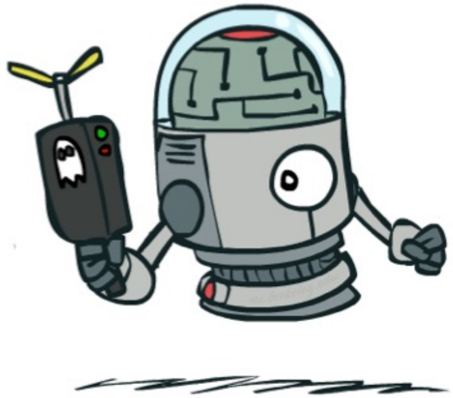
| | | | | | |
|---|---|---|---|---|---|
| 0.05 | 0.01 | 0.05 | <0.01 | <0.01 | <0.01 |
| 0.02 | 0.14 | 0.11 | 0.35 | <0.01 | <0.01 |
| 0.07 | 0.03 | 0.05 | <0.01 | 0.03 | <0.01 |
| 0.03 | 0.03 | <0.01 | <0.01 | <0.01 | <0.01 |

Before observation

| | | | | | |
|---|---|---|---|---|---|
| <0.01 | <0.01 | <0.01 | <0.01 | 0.02 | <0.01 |
| <0.01 | <0.01 | <0.01 | 0.83 | 0.02 | <0.01 |
| <0.01 | <0.01 | 0.11 | <0.01 | <0.01 | <0.01 |
| <0.01 | <0.01 | <0.01 | <0.01 | <0.01 | <0.01 |

After observation

$$B(X) \propto P(e|X)B'(X)$$



# Putting it All Together: The Forward Algorithm

- We are given evidence at each time and want to know

$$B_t(X) = P(X_t|e_{1:t})$$

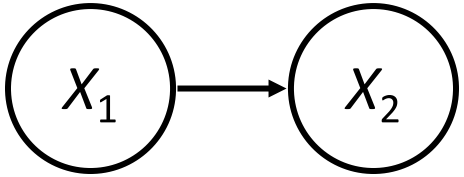
- We can derive the following updates

$$
\begin{aligned}
P(x_t|e_{1:t}) \propto_X & \; P(x_t, e_{1:t}) \\
&= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t}) \\
&= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t) \\
&= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) P(x_{t-1}, e_{1:t-1})
\end{aligned}
$$

We can normalize as we go if we want to have P(x|e) at each time step, or just once at the end…

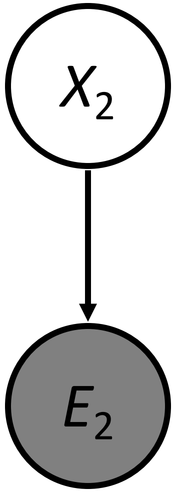
# Online Belief Updates

- Every time step, we start with current P(X | evidence)
- We update for time:




$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$

- We update for evidence:

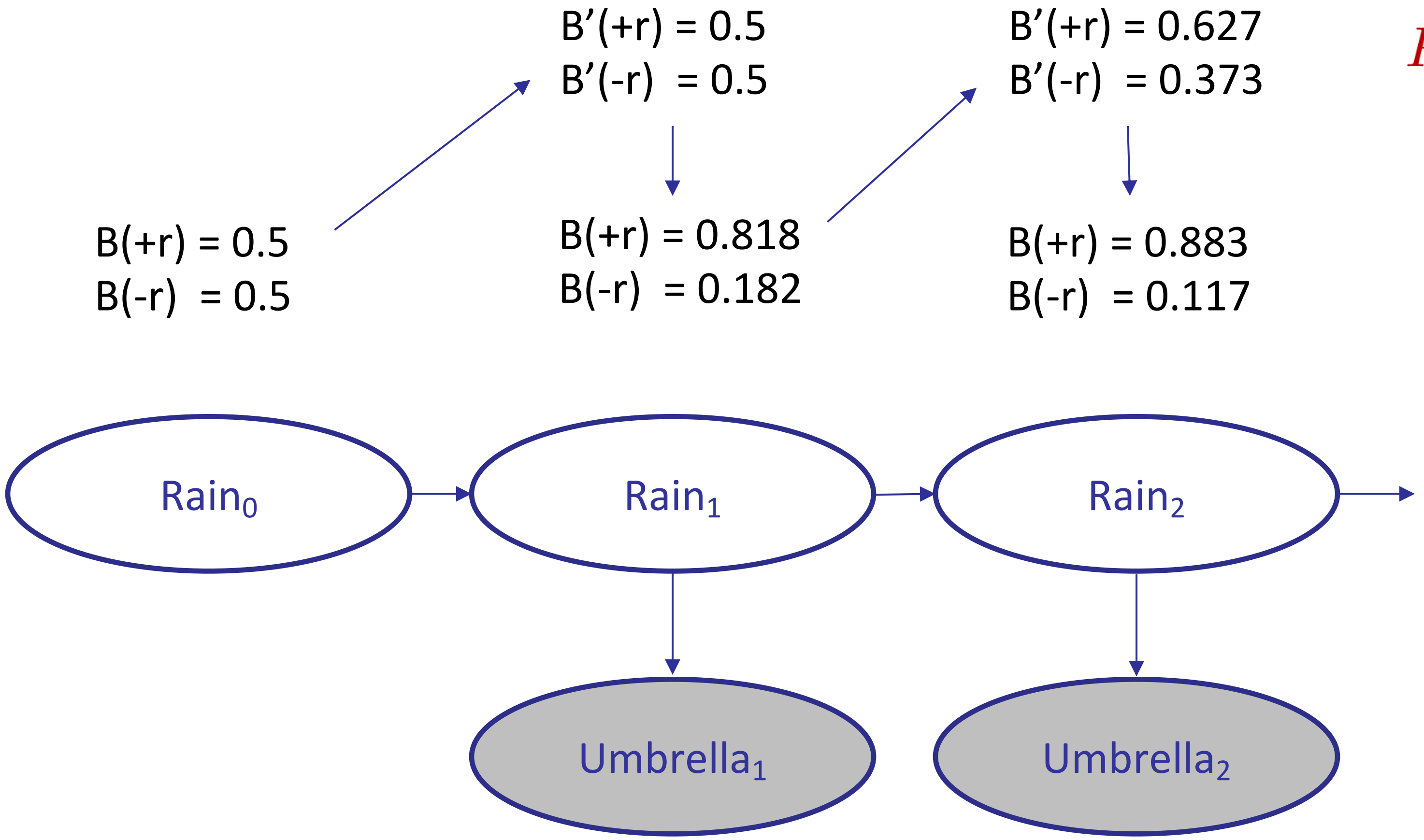



$$P(x_t|e_{1:t}) \propto_X P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$

- The forward algorithm does both at once (and doesn't normalize)

# Example: Weather HMM

$$\color{red}{P(x_t|e_{1:t})} \propto_X \color{green}{P(x_t|e_{1:t-1})} \cdot P(e_t|x_t)$$




| $R_t$ | $R_{t+1}$ | $P(R_{t+1}\|R_t)$ |
|---|---|---|
| +r | +r | 0.7 |
| +r | -r | 0.3 |
| -r | +r | 0.3 |
| -r | -r | 0.7 |

| $R_t$ | $U_t$ | $P(U_t\|R_t)$ |
|---|---|---|
| +r | +u | 0.9 |
| +r | -u | 0.1 |
| -r | +u | 0.2 |
| -r | -u | 0.8 |

# Exercises: Hidden Markov Models

Consider a Markov Model with a binary state $X$ (i.e., $X_t$ is either 0 or 1). The transition probabilities are given as follows:

| $X_t$ | $X_{t+1}$ | $P(X_{t+1} \mid X_t)$ |
|---|---|---|
| 0 | 0 | 0.9 |
| 0 | 1 | 0.1 |
| 1 | 0 | 0.5 |
| 1 | 1 | 0.5 |

**(a)** **(2 pt)** The prior belief distribution over the initial state $X_0$ is uniform, i.e., $P(X_0 = 0) = P(X_0 = 1) = 0.5$. After one timestep, what is the new belief distribution, $P(X_1)$?

# Exercises: Hidden Markov Models

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| $X_t$ | $X_{t+1}$ | $P(X_{t+1} \mid X_t)$ |
|---|---|---|
| 0 | 0 | 0.9 |
| 0 | 1 | 0.1 |
| 1 | 0 | 0.5 |
| 1 | 1 | 0.5 |

Now, we incorporate sensor readings. The sensor model is parameterized by a number $\beta \in [0, 1]$:

| $X_t$ | $E_t$ | $P(E_t \mid X_t)$ |
|---|---|---|
| 0 | 0 | $\beta$ |
| 0 | 1 | $(1 - \beta)$ |
| 1 | 0 | $(1 - \beta)$ |
| 1 | 1 | $\beta$ |

**(b) (2 pt)** At $t = 1$, we get the first sensor reading, $E_1 = 0$. Use your answer from part (a) to compute $P(X_1 = 0 \mid E_1 = 0)$. Leave your answer in terms of $\beta$.

# Exercises: Hidden Markov Models

Consider a Markov Model with a binary state $X$ (i.e., $X_t$ is either 0 or 1). The transition probabilities are given as follows:

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| 0 | 1 | 0.1 |
| 1 | 0 | 0.5 |
| 1 | 1 | 0.5 |

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| $X_t$ | $E_t$ | $P(E_t \mid X_t)$ |
|---|---|---|
| 0 | 0 | $\beta$ |
| 0 | 1 | $(1-\beta)$ |
| 1 | 0 | $(1-\beta)$ |
| 1 | 1 | $\beta$ |

**(b) (2 pt)** At $t = 1$, we get the first sensor reading, $E_1 = 0$. Use your answer from part (a) to compute $P(X_1 = 0 \mid E_1 = 0)$. Leave your answer in terms of $\beta$.

**(d) (2 pt)** Unfortunately, the sensor breaks after just one reading, and we receive no further sensor information. Compute $P(X_\infty \mid E_1 = 0)$, the stationary distribution *very many* timesteps from now.

# Next Time: Particle Filters