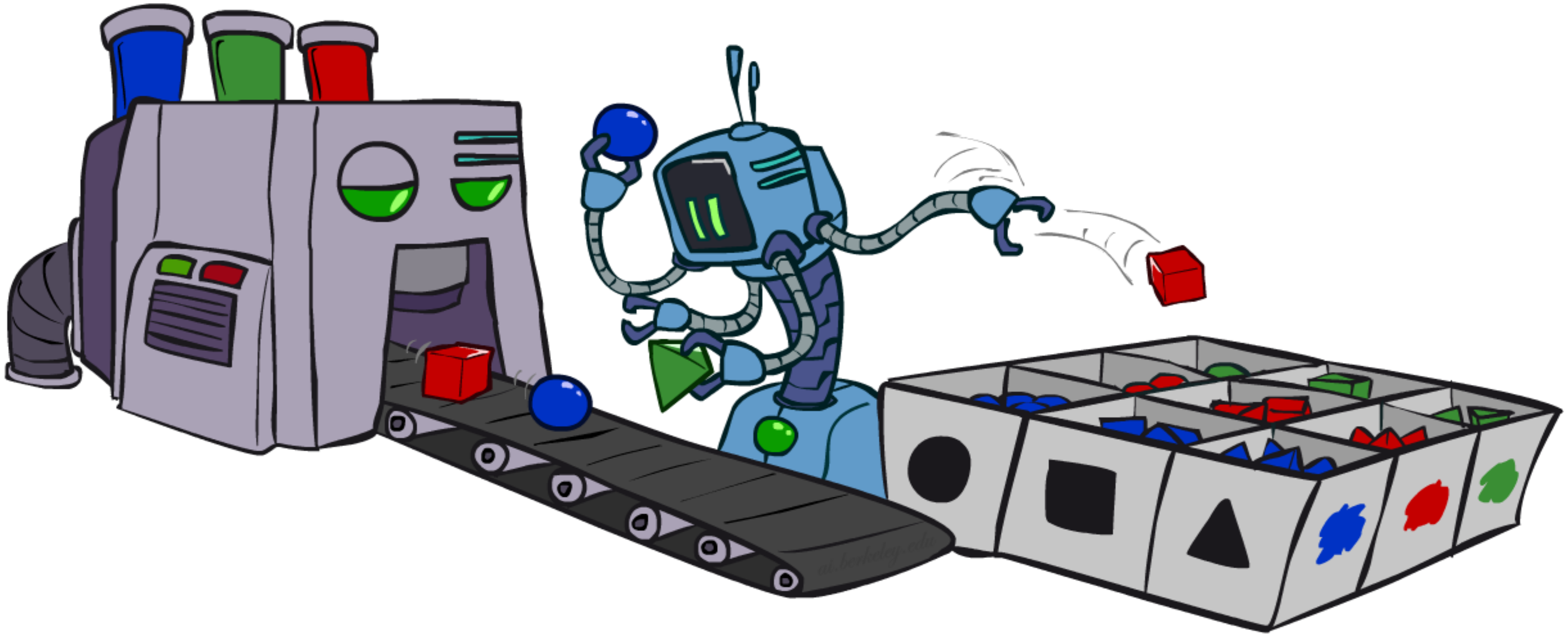
CS 343: Artificial Intelligence

Bayes Nets: Sampling



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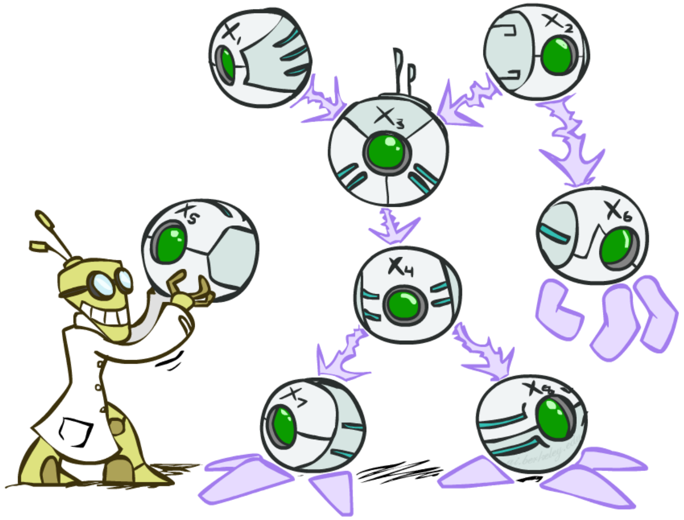
[These slides based on those of Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley. All CS188 materials are available at http://ai.berkeley.edu.]

Bayes Net Representation

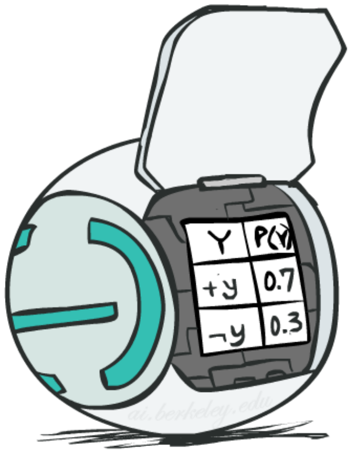
- A directed, acyclic graph, one node per random variable
- A conditional probability table (CPT) for each node
  - A collection of distributions over X, one for each combination of parents' values

$$P(X|a_1 \ldots a_n)$$



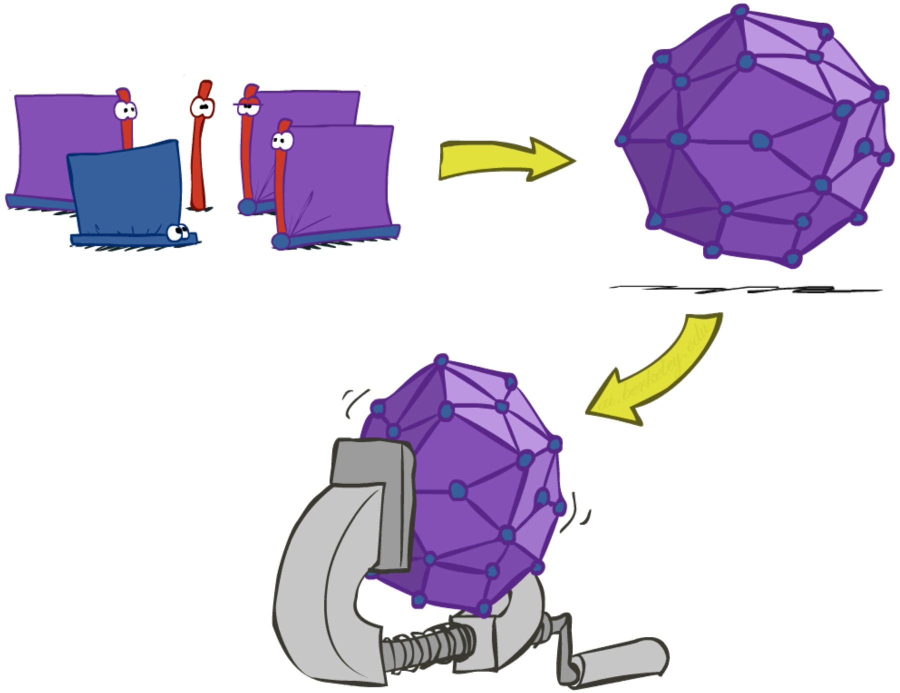
- Bayes nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i|\mathit{parents}(X_i))$$

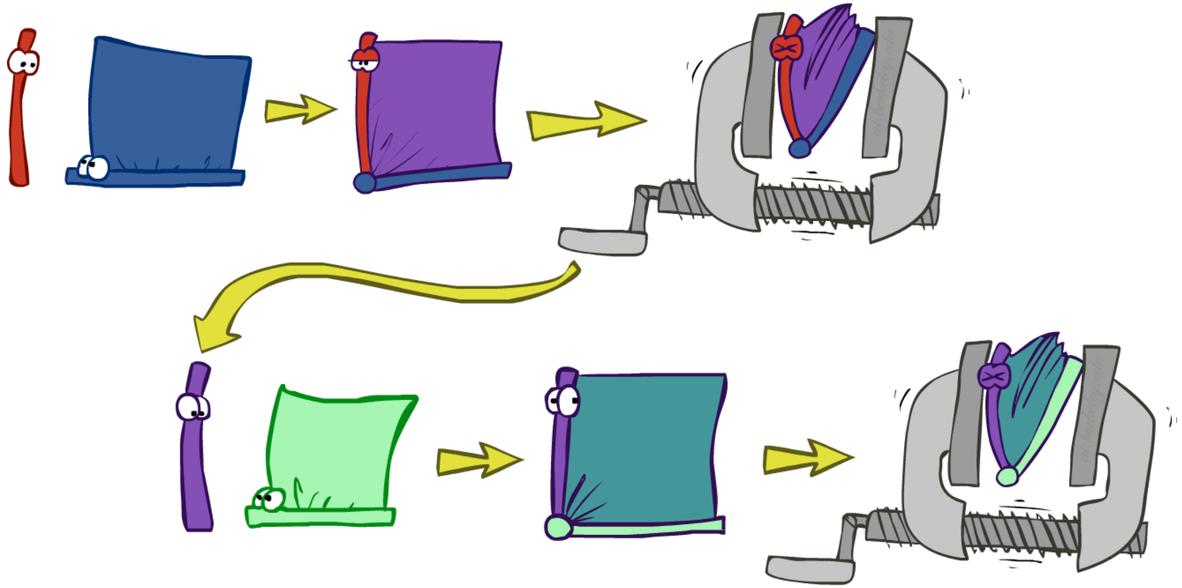


Inference by Enumeration vs. Variable Elimination

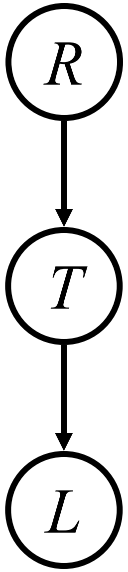
- Why is inference by enumeration so slow?
  - You join up the whole joint distribution before you sum out the hidden variables



- Idea: interleave joining and marginalizing!
  - Called "Variable Elimination"
  - Still NP-hard, but usually much faster than inference by enumeration



Traffic Domain



$$P(L) = \,?$$

- Inference by Enumeration

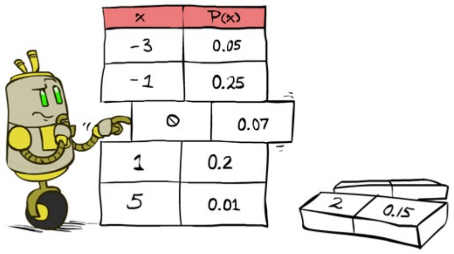
$$= \underbrace{\sum_t \underbrace{\sum_r \underbrace{P(L|t)\underbrace{P(r)P(t|r)}_{\text{Join on r}}}_{\text{Join on t}}}_{\text{Eliminate r}}}_{\text{Eliminate t}}$$

- Variable Elimination

$$= \underbrace{\sum_t \underbrace{P(L|t) \underbrace{\sum_r \underbrace{P(r)P(t|r)}_{\text{Join on r}}}_{\text{Eliminate r}}}_{\text{Join on t}}}_{\text{Eliminate t}}$$

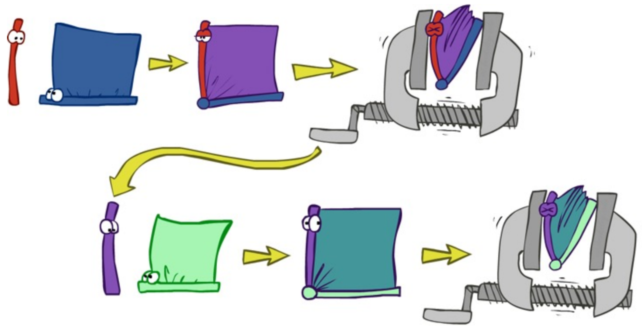
General Variable Elimination

- Query: $$P(Q|E_1 = e_1, \dots E_k = e_k)$$

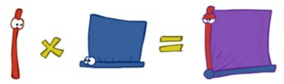


- Start with initial factors:
  - Local CPTs (but instantiated by evidence)

- While there are still hidden variables (not Q or evidence):
  - Pick a hidden variable H
  - Join all factors mentioning H
  - Eliminate (sum out) H



- Join all remaining factors and normalize



$$\times \frac{1}{Z}$$

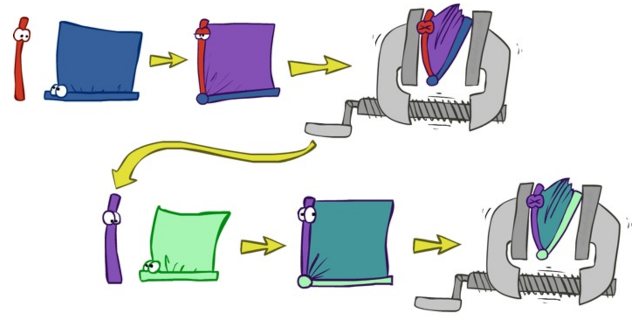
Variable Elimination Efficiency

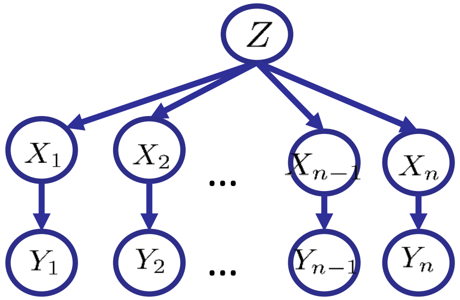
- Interleave joining and marginalizing, instead of fully joining all at once (i.e. enumeration)

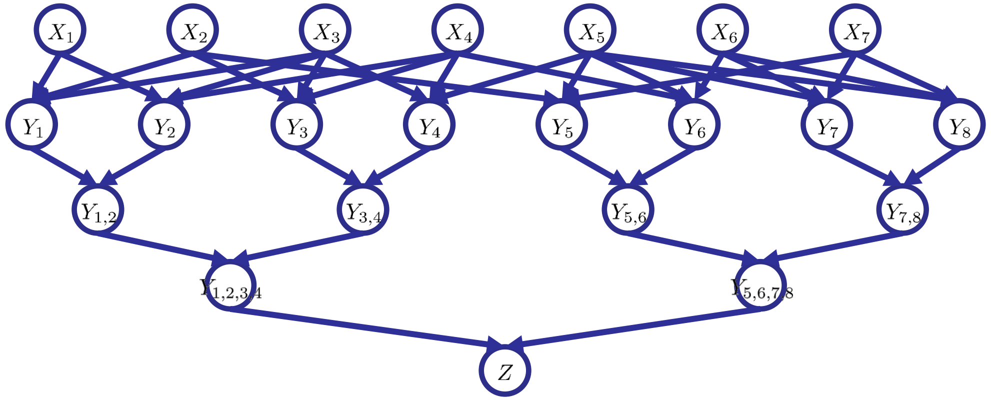
- d^k entries computed for a factor over k variables with domain sizes d

- Ordering of elimination of hidden variables can affect size of factors generated

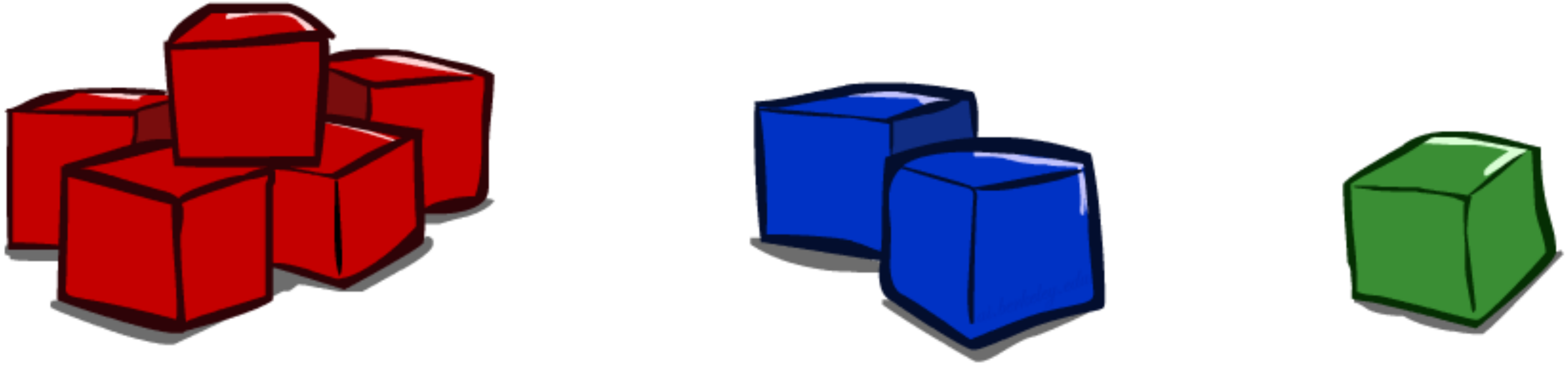
- Worst case: running time exponential in the size of the Bayes net (NP-hard)







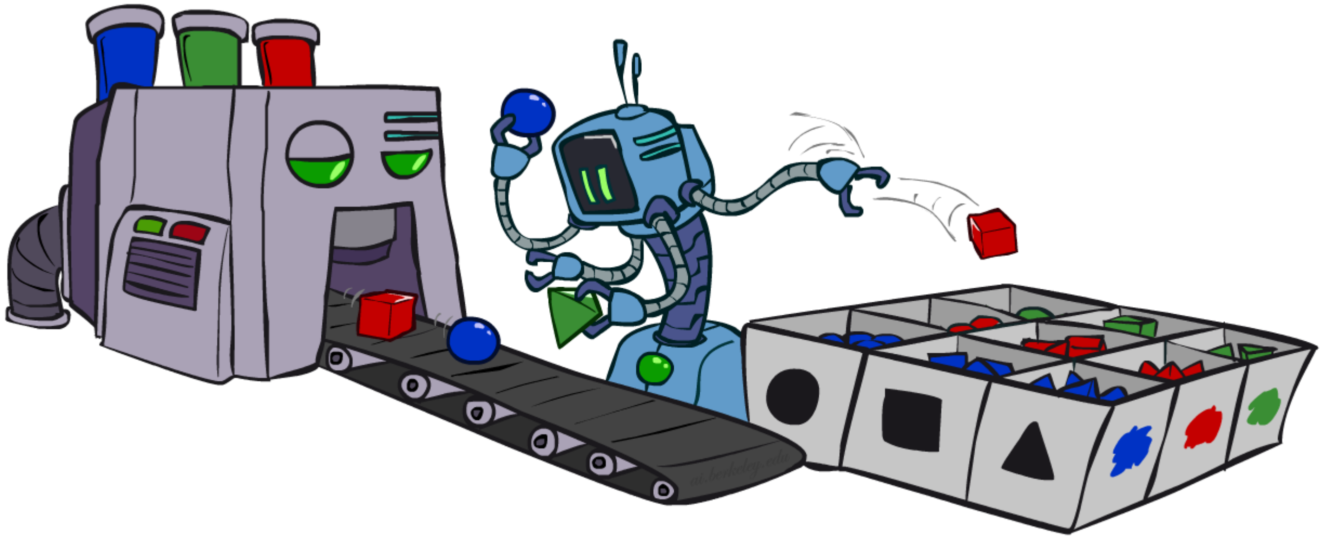
Approximate Inference: Sampling



Sampling

- Basic idea
  - Draw N samples from a sampling distribution S
  - Compute an approximate posterior probability
  - Show this converges to the true probability P

- Why sample?
  - Learning: get samples from a distribution you don't know
  - Inference: getting samples can be faster than computing the right answer (e.g. with variable elimination)



Sampling

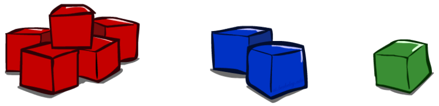
- Sampling from given distribution
  - Step 1: Get sample *u* from uniform distribution over [0, 1)
    - E.g. random() in python
  - Step 2: Convert this sample *u* into an outcome for the given distribution by having each outcome associated with a sub-interval of [0,1) with sub-interval size equal to probability of the outcome

- Example

C	P(C)
red	0.6
green	0.1
blue	0.3

$$0 \le u < 0.6, \rightarrow C = red$$
$$0.6 \le u < 0.7, \rightarrow C = green$$
$$0.7 \le u < 1, \rightarrow C = blue$$

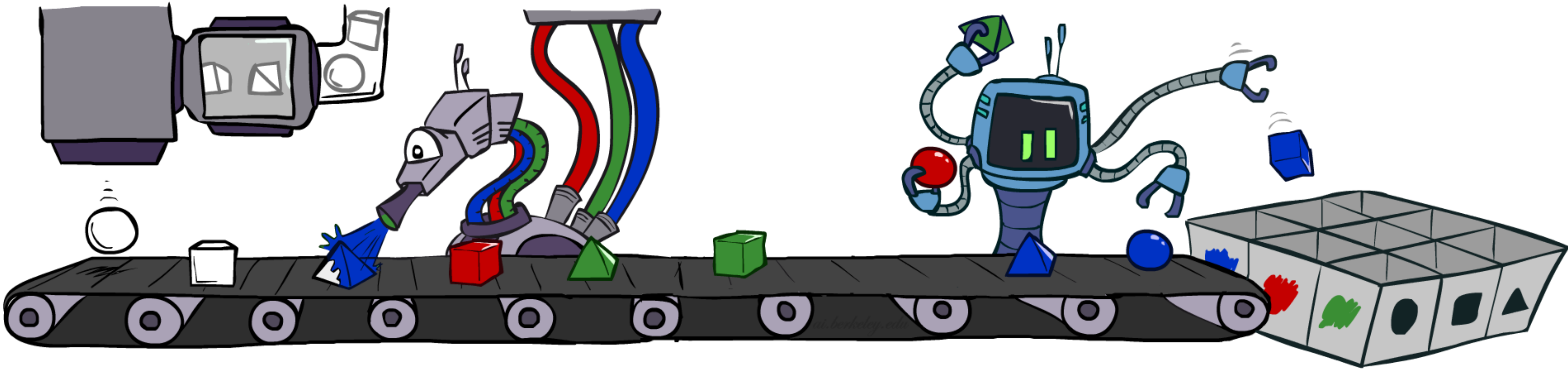
- If random() returns *u* = 0.83, then our sample is *C* = blue
- E.g, after sampling 8 times:



Sampling in Bayes Nets

- Prior Sampling
- Rejection Sampling
- Likelihood Weighting
- Gibbs Sampling

Prior Sampling



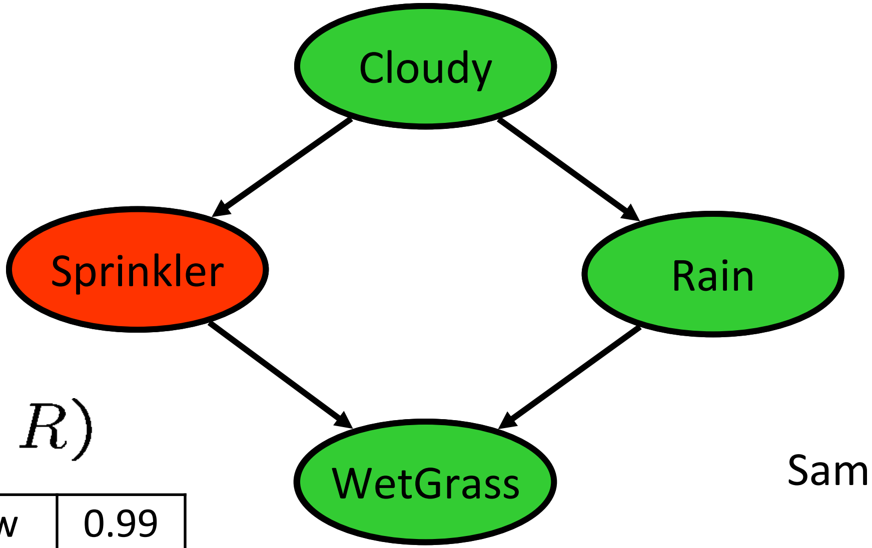
Prior Sampling

$P(C)$

+c	0.5
-c	0.5

$P(S|C)$

+c	+s	0.1
	-s	0.9
-c	+s	0.5
	-s	0.5



$P(R|C)$

+c	+r	0.8
	-r	0.2
-c	+r	0.2
	-r	0.8

$P(W|S,R)$

+s	+r	+w	0.99
		-w	0.01
	-r	+w	0.90
		-w	0.10
-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

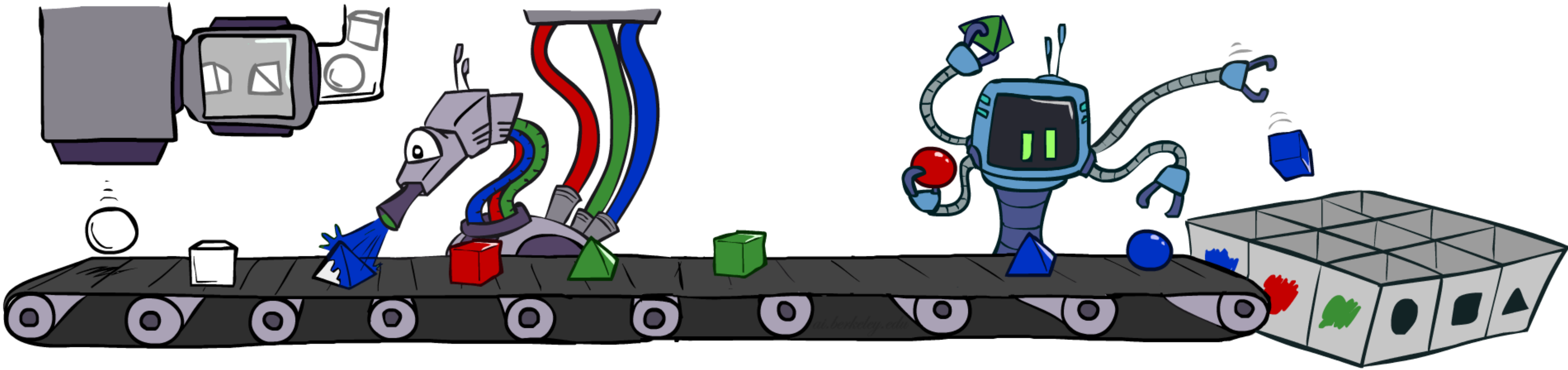
+c, -s, +r, +w

-c, +s, -r, +w

…

Prior Sampling

- For i=1, 2, …, n
  - Sample x_i from $P(X_i \mid Parents(X_i))$
- Return $(x_1, x_2, \ldots, x_n)$



Prior Sampling

- This process generates samples with probability:

$$S_{PS}(x_1 \dots x_n) = \prod_{i=1}^{n} P(x_i|\text{Parents}(X_i)) = P(x_1 \dots x_n)$$

…i.e. the BN's joint probability

- Let the number of samples of a particular event be $N_{PS}(x_1 \dots x_n)$ and the total number of samples of all events be N.
- Then
$$\begin{aligned}\lim_{N\to\infty} \widehat{P}(x_1,\dots,x_n) &= \lim_{N\to\infty} N_{PS}(x_1,\dots,x_n)/N \\ &= S_{PS}(x_1,\dots,x_n) \\ &= P(x_1 \dots x_n)\end{aligned}$$
- I.e., the sampling procedure is consistent

Example

- We'll get a bunch of samples from the BN:

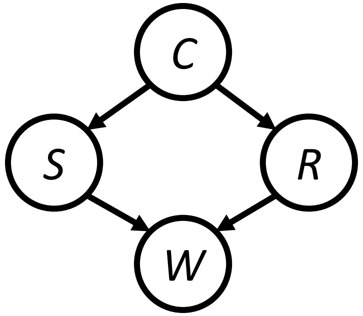
  +c, -s, +r, +w

  +c, +s, +r, +w

  -c, +s, +r, -w

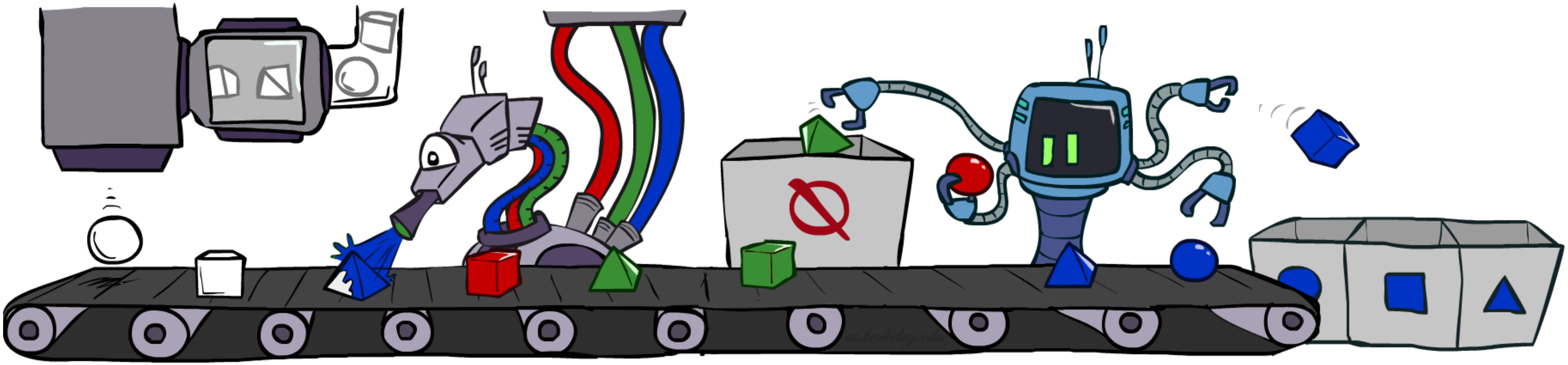
  -c, -s, +r, +w

  +c, -s, -r, +w



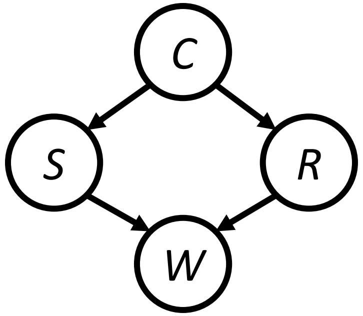
- If we want to know P(W)
  - We have counts <+w:4, -w:1>
  - Normalize to get P(W) = <+w:0.8, -w:0.2>
  - This will get closer to the true distribution with more samples
  - Can estimate anything else, too
  - What about P(C | +w)?   P(C | +r, +w)? P(C | -r, -w)?
  - Fast: can use fewer samples if less time (what's the drawback?)

Rejection Sampling



Rejection Sampling

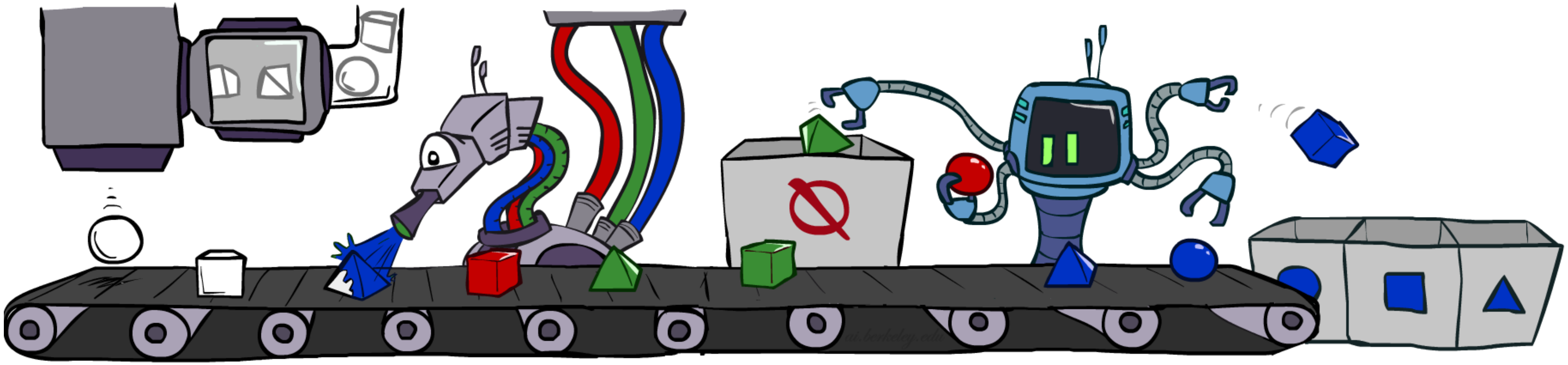
- Let’s say we want P(C)
  - No point keeping all samples around
  - Just tally counts of C as we go
- Let’s say we want P(C | +s)
  - Same thing: tally C outcomes, but ignore (reject) samples which don’t have S=+s
  - This is called rejection sampling
  - It is also consistent for conditional probabilities (i.e., correct in the limit)



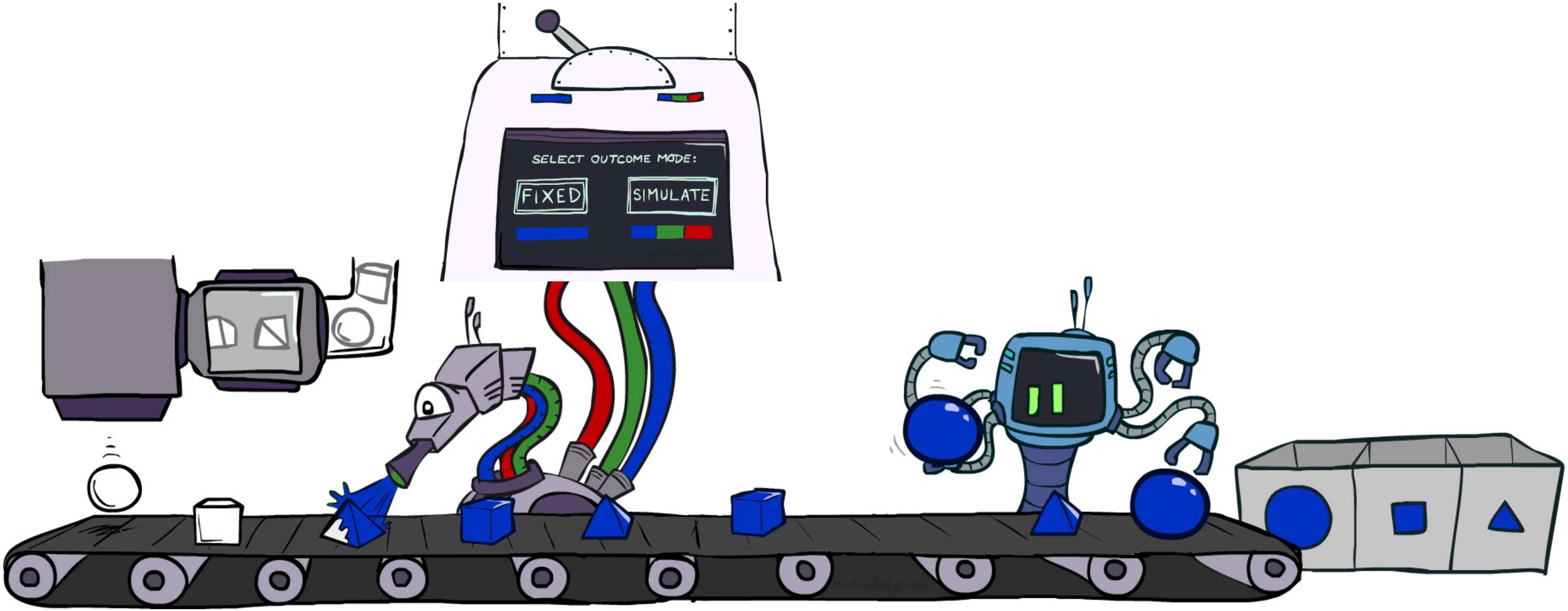
~~+c, -s, +r, +w~~
+c, +s, +r, +w
-c, +s, +r, -w
~~+c, -s, +r, +w~~
~~-c, -s, -r, +w~~

Rejection Sampling

- IN: evidence instantiation
- For i=1, 2, …, n
  - Sample x_i from $P(X_i \mid Parents(X_i))$
  - If x_i not consistent with evidence
    - Reject: Return, and no sample is generated in this cycle
- Return $(x_1, x_2, \ldots, x_n)$

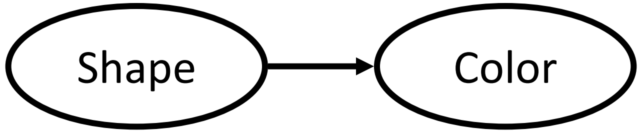


Likelihood Weighting

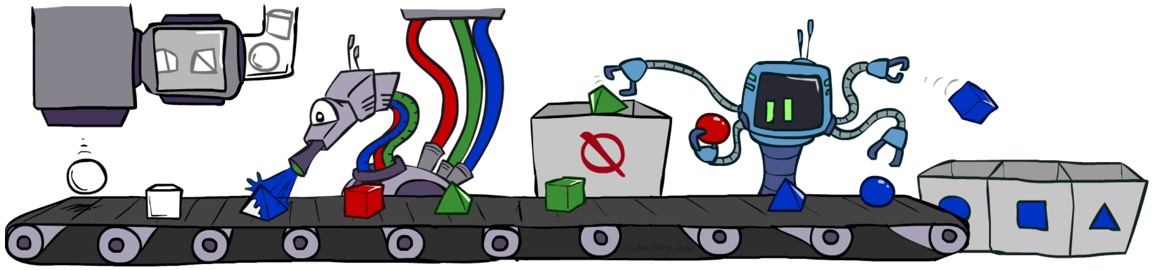


Likelihood Weighting

- Problem with rejection sampling:
  - If evidence is unlikely, rejects lots of samples
  - Evidence not exploited as you sample
  - Consider P(Shape | blue)

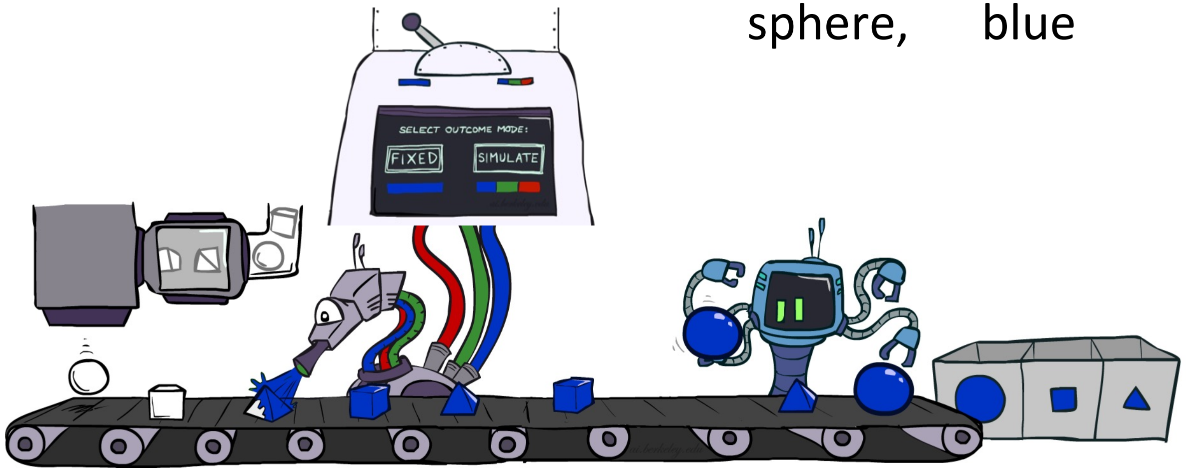


~~pyramid, green~~
~~pyramid, red~~
sphere, blue
~~cube, red~~
~~sphere, green~~



- Idea: fix evidence variables and sample the rest
  - Problem: sample distribution not consistent!
  - Solution: weight by probability of evidence given parents

pyramid, blue
pyramid, blue
sphere, blue
cube, blue
sphere, blue



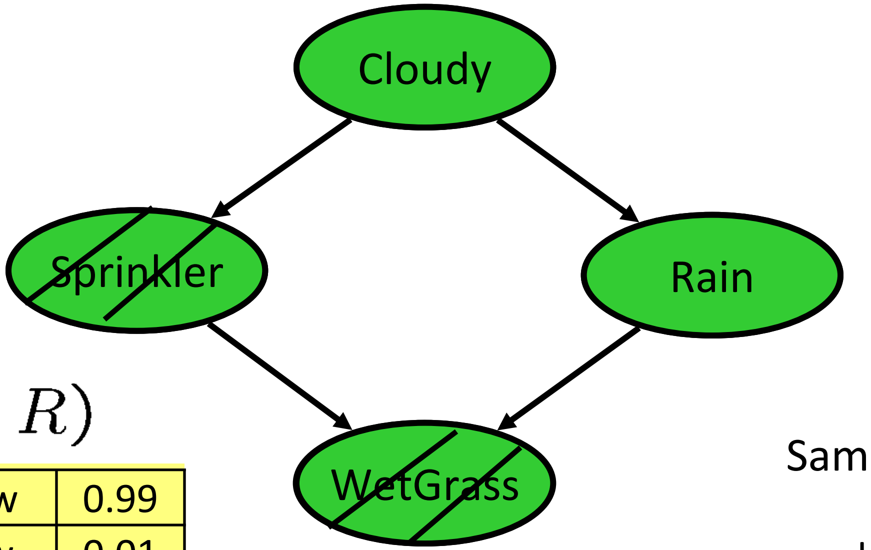
Likelihood Weighting

$P(C)$

+c	0.5
-c	0.5

$P(S|C)$

+c	+s	0.1
	-s	0.9
-c	+s	0.5
	-s	0.5



$P(R|C)$

+c	+r	0.8
	-r	0.2
-c	+r	0.2
	-r	0.8

$P(W|S,R)$

+s	+r	+w	0.99
		-w	0.01
	-r	+w	0.90
		-w	0.10
-s	+r	+w	0.90
		-w	0.10
	-r	+w	0.01
		-w	0.99

Samples:

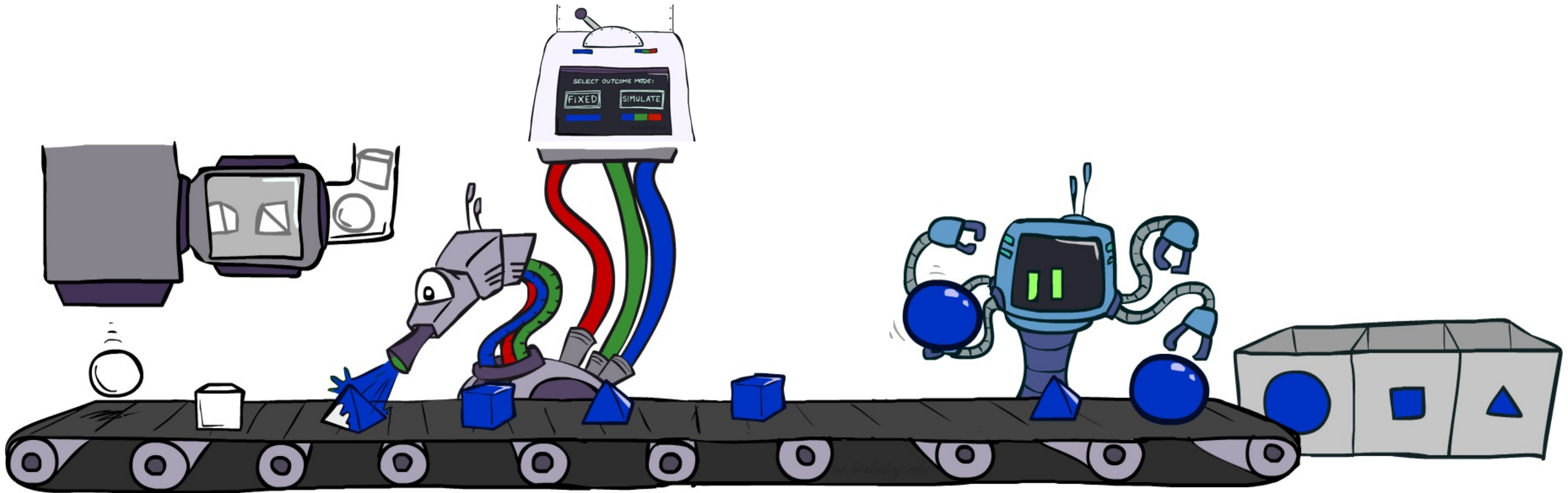
+c, +s, +r, +w

…

$$w = 1.0 \times 0.1 \times 0.99$$

Likelihood Weighting

- IN: evidence instantiation
- $w = 1.0$
- for $i=1, 2, \ldots, n$
  - if X_i is an evidence variable
    - X_i = observation x_i for X_i
    - Set $w = w * P(x_i \mid \text{Parents}(X_i))$
  - else
    - Sample x_i from $P(X_i \mid \text{Parents}(X_i))$
- return $(x_1, x_2, \ldots, x_n)$, w



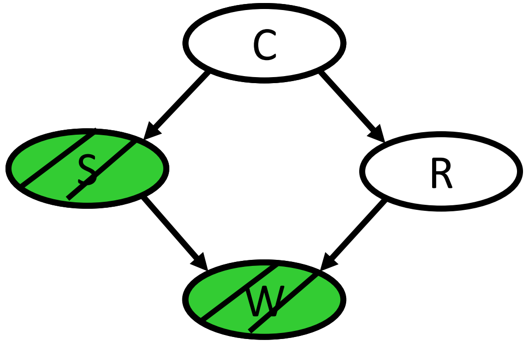
Likelihood Weighting

- Sampling distribution if z sampled and e fixed evidence

$$S_{WS}(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{l} P(z_i|\text{Parents}(Z_i))$$

- Now, samples have weights

$$w(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{m} P(e_i|\text{Parents}(E_i))$$



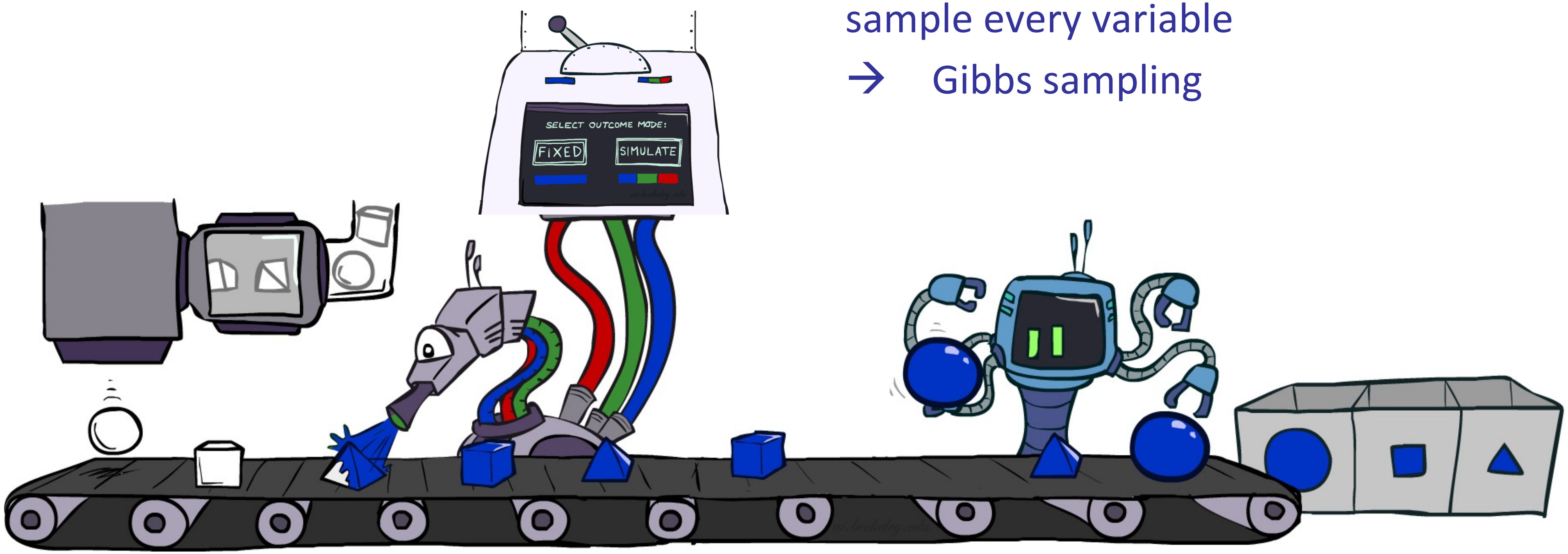
- Together, weighted sampling distribution is consistent

$$S_{\text{WS}}(z, e) \cdot w(z, e) = \prod_{i=1}^{l} P(z_i|\text{Parents}(z_i)) \prod_{i=1}^{m} P(e_i|\text{Parents}(e_i))$$

$$= P(\mathbf{z}, \mathbf{e})$$

Likelihood Weighting

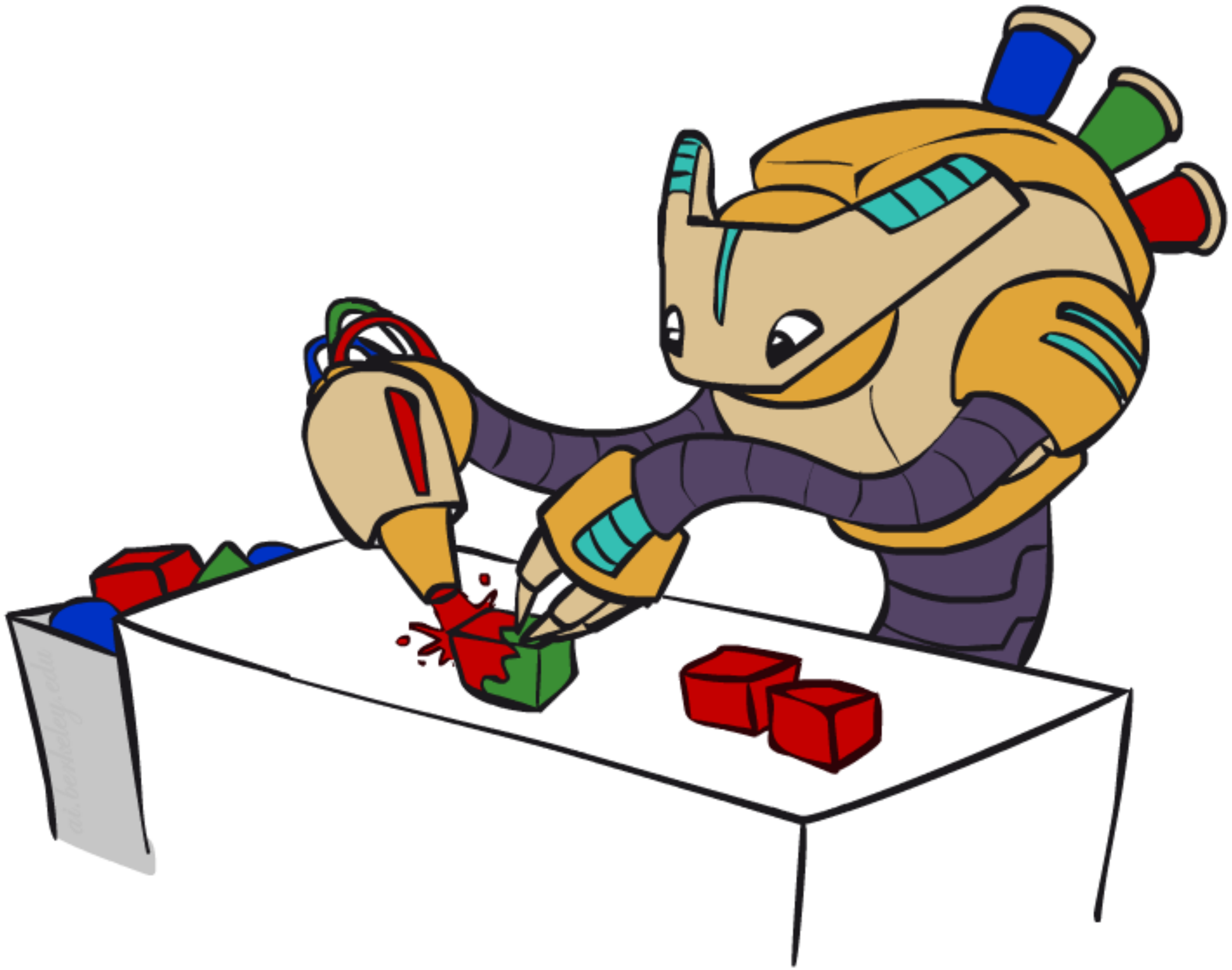
- Likelihood weighting is good
  - We have taken evidence into account as we generate the sample
  - Our samples will reflect the state of the world suggested by the evidence
  - No need for rejection!



- Likelihood weighting doesn't solve all our problems
  - Evidence influences the choice of downstream variables, but not upstream ones (not more likely to get a value matching the evidence)
  - Can cause many very small weights —> inefficient!
- We would like to consider evidence when we sample every variable

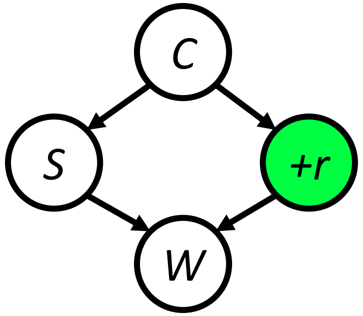
→ Gibbs sampling

Gibbs Sampling

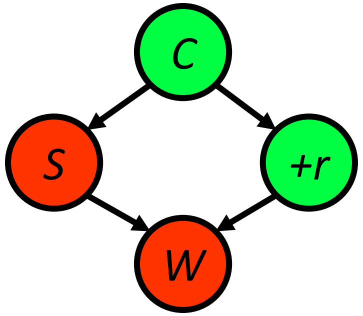


Gibbs Sampling Example: P(S | +r)

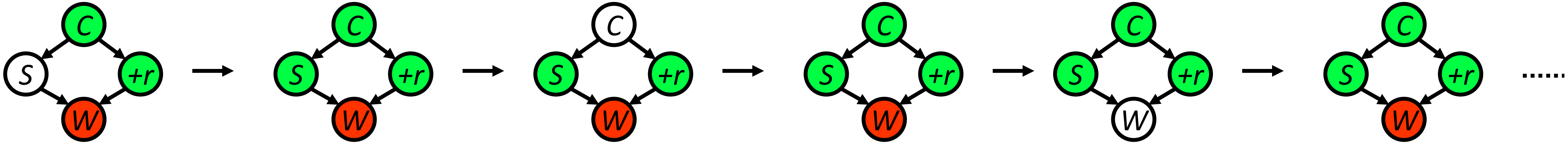
- Step 1: Fix evidence
  - R = +r



- Step 2: Initialize other variables
  - Randomly



- Step 3: Repeat the following:
  - Choose a non-evidence variable X
  - Resample X from P(X | all other variables)



Sample from $P(S|+c,-w,+r)$ Sample from $P(C|+s,-w,+r)$ Sample from $P(W|+s,+c,+r)$

Gibbs Sampling

- How is this better than sampling from the full joint?
  - In a Bayes Net, sampling a variable given all the other variables (e.g. P(R|S,C,W)) is usually much easier than sampling from the full joint distribution
  - Only requires one join on the variable to be sampled (in this case, a join on R)

Further Reading on Gibbs Sampling*

- Gibbs sampling produces sample from the query distribution P(Q | e) in limit of re-sampling infinitely often

- Gibbs sampling is a special case of more general methods called Markov chain Monte Carlo (MCMC) methods

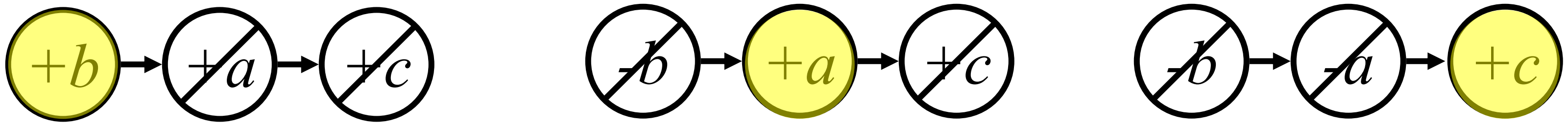
  - Metropolis-Hastings is one of the more famous MCMC methods (in fact, Gibbs sampling is a special case of Metropolis-Hastings)

- You may read about Monte Carlo methods – they're just sampling

Markov Chain Monte Carlo*

- *Idea*: instead of sampling from scratch, create samples that are each like the last one.

- *Procedure*: resample one variable at a time, conditioned on all the rest, but keep evidence fixed. E.g., for P(b|c):

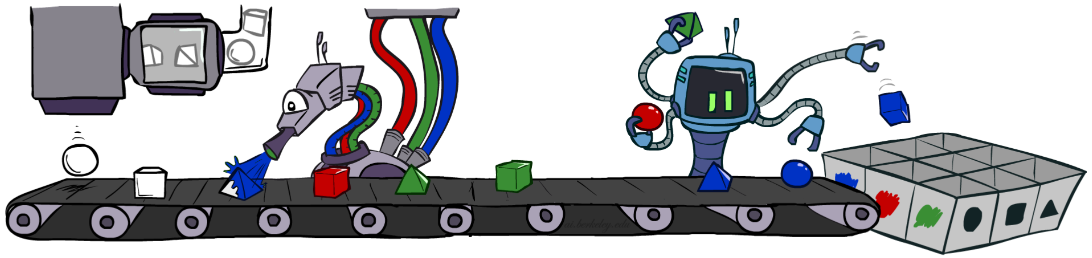


- *Properties*: Now samples are not independent (in fact they're nearly identical), but sample averages are still consistent estimators!

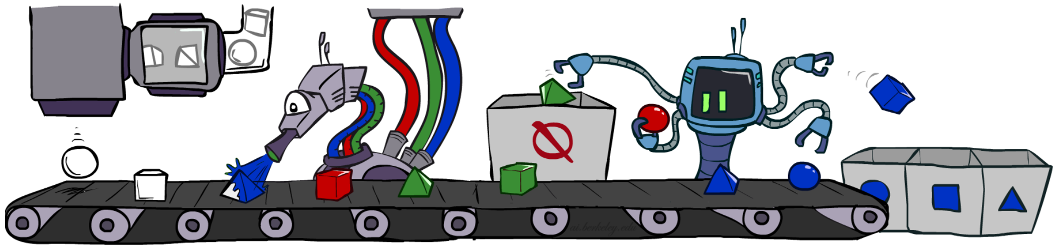
- *What's the point*: both upstream and downstream variables condition on evidence.

Bayes Net Sampling Summary

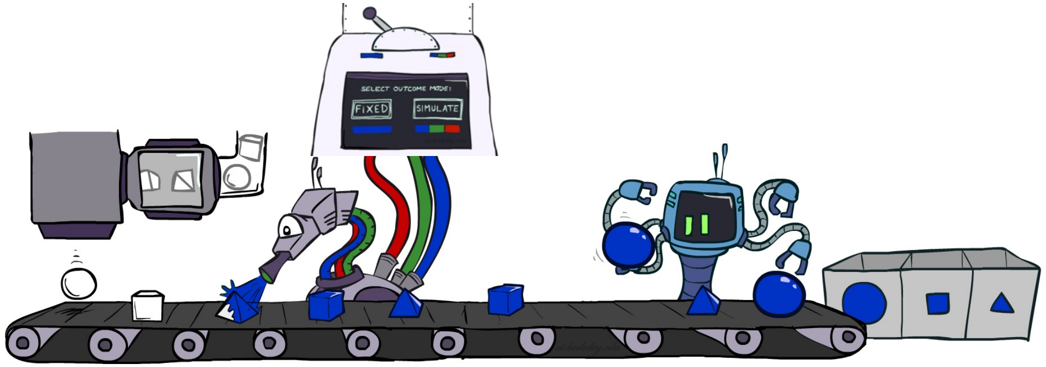
- Prior Sampling P(Q)



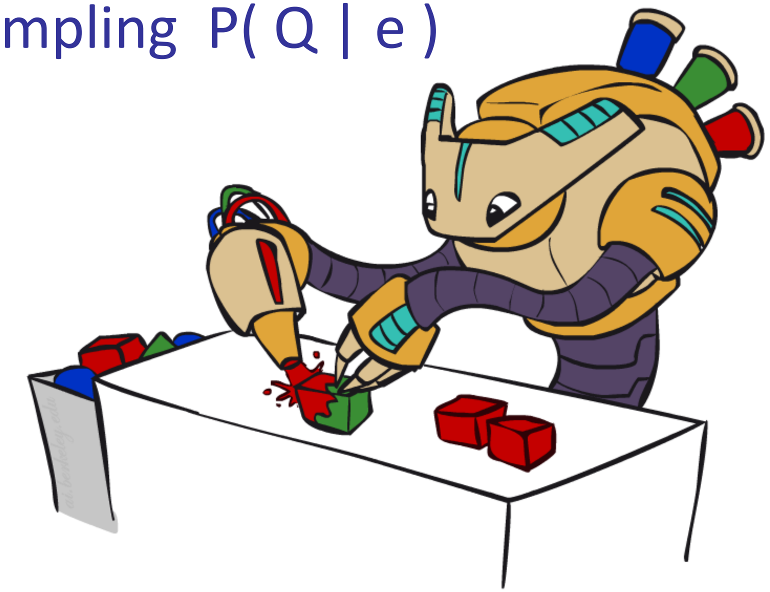
- Rejection Sampling P(Q | e)



- Likelihood Weighting P(Q | e)



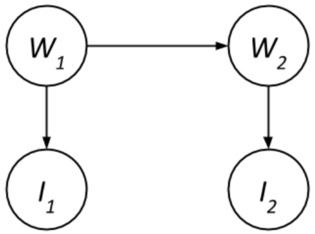
- Gibbs Sampling P(Q | e)



Exercise: Bayes Nets Sampling

2 Sampling and Dynamic Bayes Nets

We would like to analyze people's ice cream eating habits on sunny and rainy days Suppose we consider the weather, along with a person's ice-cream eating, over the span of two days. We'll have four random variables: W_1 and W_2 stand for the weather on days 1 and 2, which can either be rainy R or sunny S, and the variables I_1 and I_2 represent whether or not the person ate ice cream on days 1 and 2, and take values T (for truly eating ice cream) or F. We can model this as the following Bayes Net with these probabilities.



W_1	$P(W_1)$
S	0.6
R	0.4

W_1	W_2	$P(W_2 \vert W_1)$
S	S	0.7
S	R	0.3
R	S	0.5
R	R	0.5

W	I	$P(I \vert W)$
S	T	0.9
S	F	0.1
R	T	0.2
R	F	0.8

Suppose we produce the following samples of (W_1, I_1, W_2, I_2) from the ice-cream model:

R, F, R, F   R, F, R, F   S, F, S, T   S, T, S, T   S, T, R, F
R, F, R, T   S, T, S, T   S, T, S, T   S, T, R, F   R, F, S, T

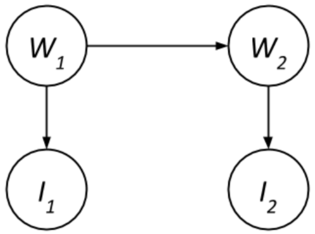
1. What is $\widehat{P}(W_2 = \mathtt{R})$, the probability that sampling assigns to the event $W_2 = \mathtt{R}$?

2. Cross off samples above which are rejected by rejection sampling if we're computing $P(W_2 | I_1 = \mathtt{T}, I_2 = \mathtt{F})$.

Exercise: Bayes Nets Sampling

2 Sampling and Dynamic Bayes Nets

We would like to analyze people's ice cream eating habits on sunny and rainy days Suppose we consider the weather, along with a person's ice-cream eating, over the span of two days. We'll have four random variables: W_1 and W_2 stand for the weather on days 1 and 2, which can either be rainy R or sunny S, and the variables I_1 and I_2 represent whether or not the person ate ice cream on days 1 and 2, and take values T (for truly eating ice cream) or F. We can model this as the following Bayes Net with these probabilities.



W_1	$P(W_1)$
S	0.6
R	0.4

W_1	W_2	$P(W_2 \vert W_1)$
S	S	0.7
S	R	0.3
R	S	0.5
R	R	0.5

W	I	$P(I \vert W)$
S	T	0.9
S	F	0.1
R	T	0.2
R	F	0.8

Rejection sampling seems to be wasting a lot of effort, so we decide to switch to likelihood weighting. Assume we generate the following six samples given the evidence $I_1 = \mathtt{T}$ and $I_2 = \mathtt{F}$:

$$(W_1, I_1, W_2, I_2) = \left\{(\mathtt{S},\mathtt{T},\mathtt{R},\mathtt{F}), (\mathtt{R},\mathtt{T},\mathtt{R},\mathtt{F}), (\mathtt{S},\mathtt{T},\mathtt{R},\mathtt{F}), (\mathtt{S},\mathtt{T},\mathtt{S},\mathtt{F}), (\mathtt{S},\mathtt{T},\mathtt{S},\mathtt{F}), (\mathtt{R},\mathtt{T},\mathtt{S},\mathtt{F})\right\}$$

3. What is the weight of the first sample (S, T, R, F) above?