CS 343: Artificial Intelligence

Bayes Nets: Independence

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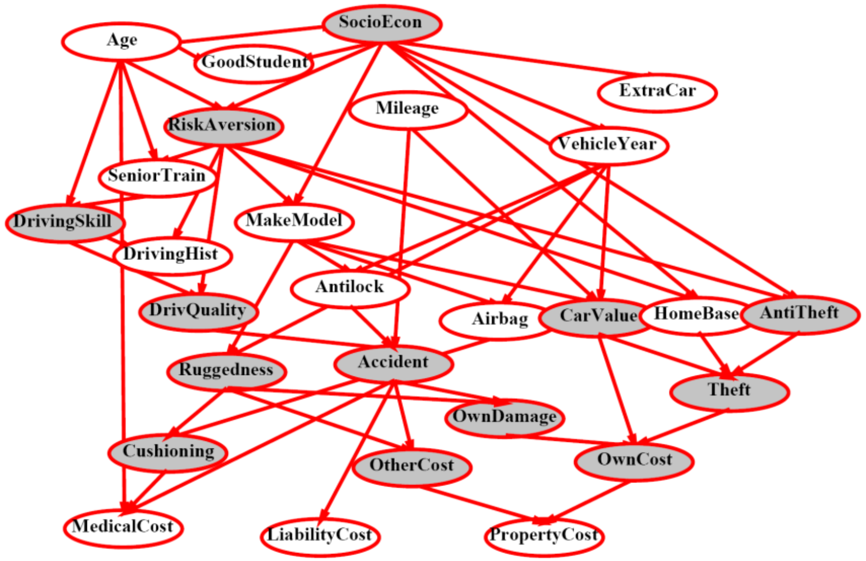
[These slides based on those of Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley. All CS188 materials are available at http://ai.berkeley.edu.]

Probability Recap

- Conditional probability
$$P(x|y) = \frac{P(x,y)}{P(y)}$$
- Product rule
$$P(x,y) = P(x|y)P(y)$$
- Chain rule
$$\begin{aligned} P(X_1, X_2, \ldots X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1,X_2)\ldots \\ &= \prod_{i=1}^{n} P(X_i|X_1,\ldots,X_{i-1}) \end{aligned}$$
- X, Y independent if and only if: $\forall x, y : P(x,y) = P(x)P(y)$
- X and Y are conditionally independent given Z if and only if:
$$\forall x, y, z : P(x,y|z) = P(x|z)P(y|z) \qquad X \perp\!\!\!\perp Y | Z$$

Bayes Nets

- A Bayes' net is an efficient encoding of a probabilistic model of a domain



- Questions we can ask:
  - Inference: given a fixed BN, what is P(X | e)?
  - Representation: given a BN graph, what kinds of distributions can it encode?
  - Modeling: what BN is most appropriate for a given domain?

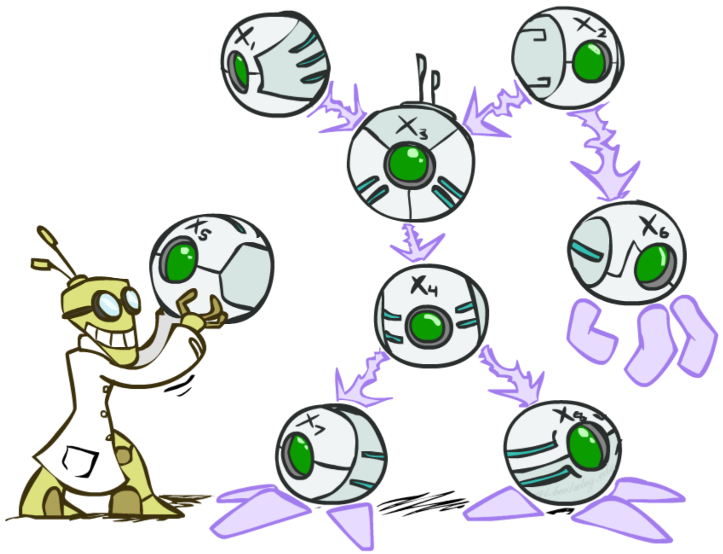
Bayes Net Semantics

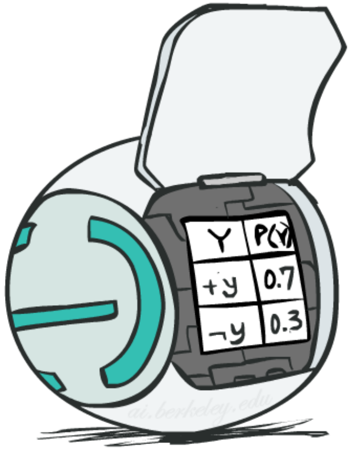
- A directed, acyclic graph, one node per random variable
- A conditional probability table (CPT) for each node
  - A collection of distributions over X, one for each combination of parents' values:

$$P(X|a_1 \ldots a_n)$$

- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

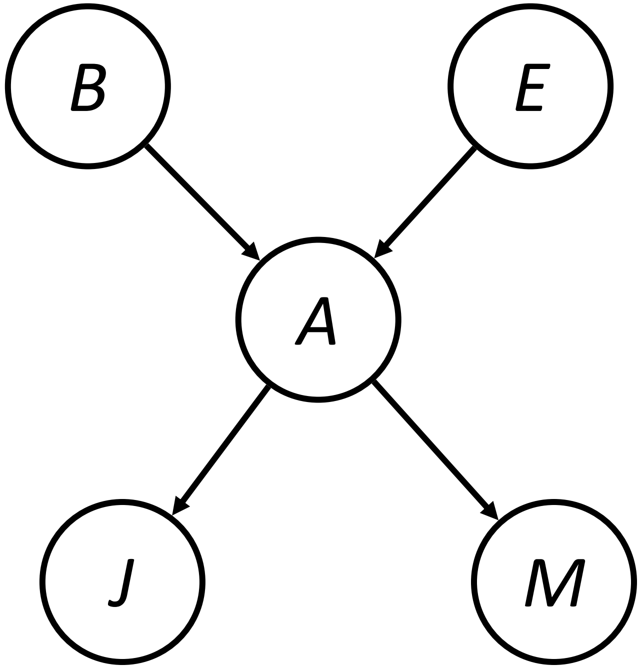
$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i|\textit{parents}(X_i))$$



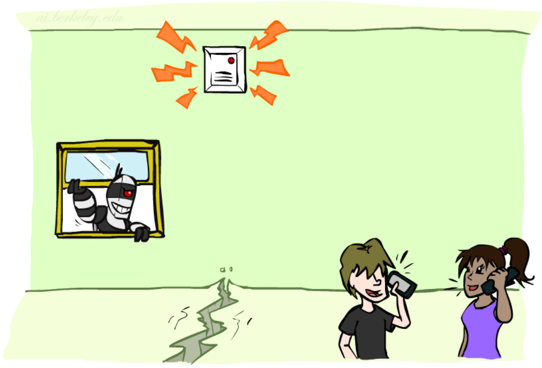


Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



E	P(E)
+e	0.002
-e	0.998



A	J	P(J|A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

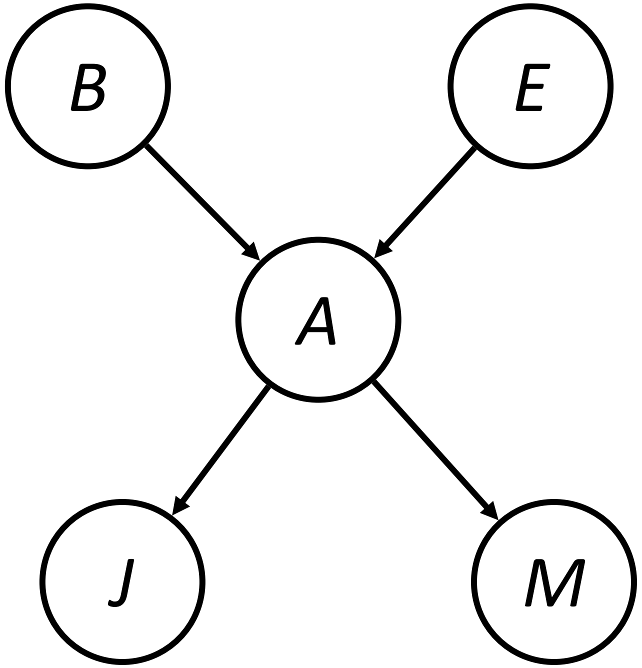
A	M	P(M|A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

$$P(+b, -e, +a, -j, +m) =$$

Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



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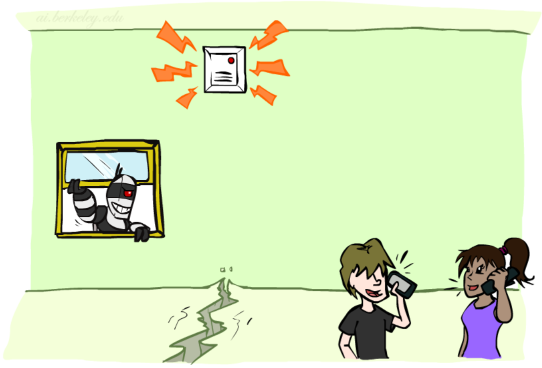

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+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
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+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

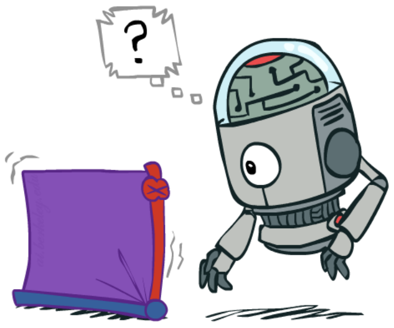
$$P(+b, -e, +a, -j, +m) =$$

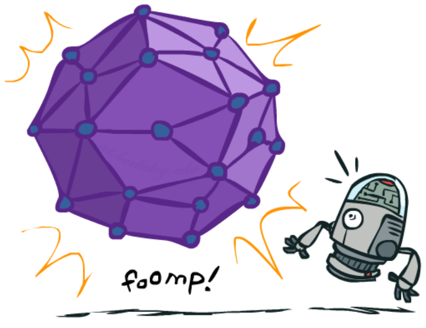
$$P(+b)P(-e)P(+a| + b, -e)P(-j| + a)P(+m| + a) =$$

$$0.001 \times 0.998 \times 0.94 \times 0.1 \times 0.7$$

Size of a Bayes Net

- How big is a joint distribution over N Boolean variables?
  - 2^N
- How big is an N-node net if nodes have up to k parents?
  - $O(N * 2^{k+1})$





- Both give you the power to calculate

$$P(X_1, X_2, \ldots X_n)$$

- BNs: Huge space savings!
- Also easier to elicit local CPTs
- Also faster to answer queries (coming)

Bayes Nets

- ✓ Representation
- Conditional Independences
- Probabilistic Inference
- Learning Bayes Nets from Data

Conditional Independence

- X and Y are **independent** if

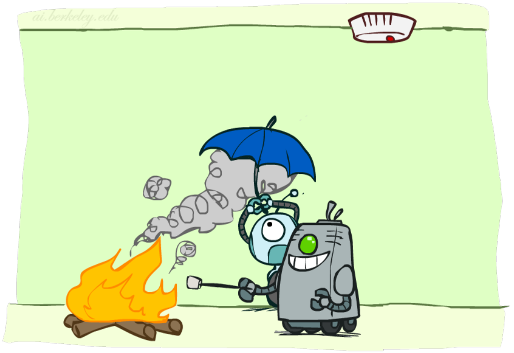
$$\forall x, y \ P(x, y) = P(x)P(y) \dashrightarrow X \perp\!\!\!\perp Y$$

- X and Y are **conditionally independent** given Z

$$\forall x, y, z \ P(x, y|z) = P(x|z)P(y|z) \dashrightarrow X \perp\!\!\!\perp Y|Z$$

- (Conditional) independence is a property of a distribution

- Example: $Alarm \perp\!\!\!\perp Fire|Smoke$

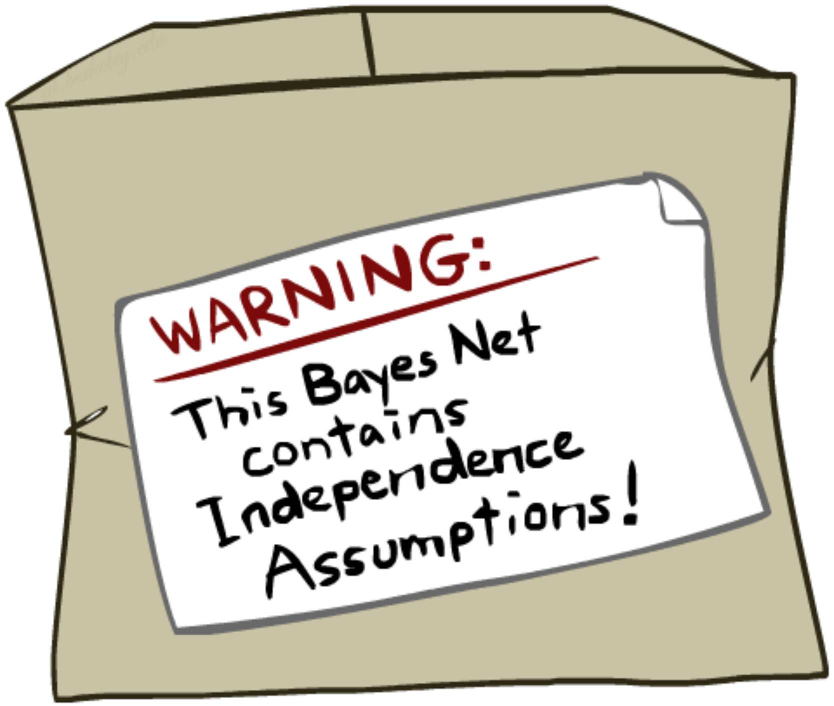


Bayes Nets: Assumptions

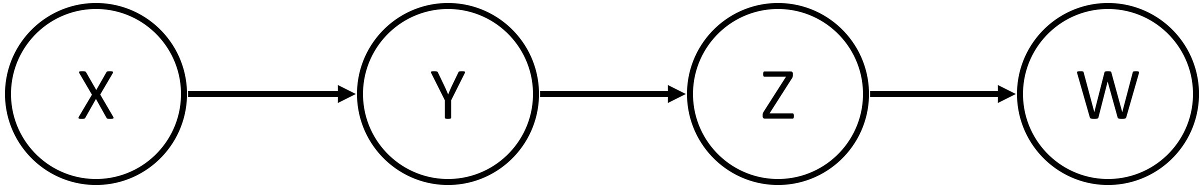
- Assumptions we are required to make to define the Bayes net when given the graph:

$$P(x_i|x_1 \cdots x_{i-1}) = P(x_i|parents(X_i))$$

- Beyond above "chain rule → Bayes net" conditional independence assumptions:
  - Often additional conditional independences
  - They can be inferred from the graph structure
- Important for modeling: understand assumptions made when choosing a Bayes net graph



Example



- Conditional independence assumptions directly from simplifications in chain rule:

Standard chain rule: $p(x, y, z, w) = p(x)p(y|x)p(z|x, y)p(w|x, y, z)$

Bayes net: $p(x, y, z, w) = p(x)p(y|x)p(z|y)p(w|z)$

Since: $z \perp\!\!\!\perp x \mid y$ and $w \perp\!\!\!\perp x, y \mid z$ (cond. indep. given parents)

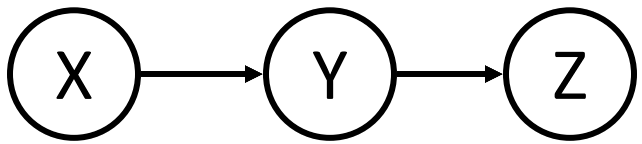
- Additional implied conditional independence assumptions? $w \perp\!\!\!\perp x \mid y$

$$p(w|x, y) = \frac{p(w, x, y)}{p(x, y)} = \frac{\sum_z p(x)p(y|x)p(z|y)p(w|z)}{p(x)p(y|x)} = \sum_z p(z|y)p(w|z) = \sum_z p(z|y)p(w|z, y)$$

$$= \sum_z p(z, w|y) = p(w|y)$$

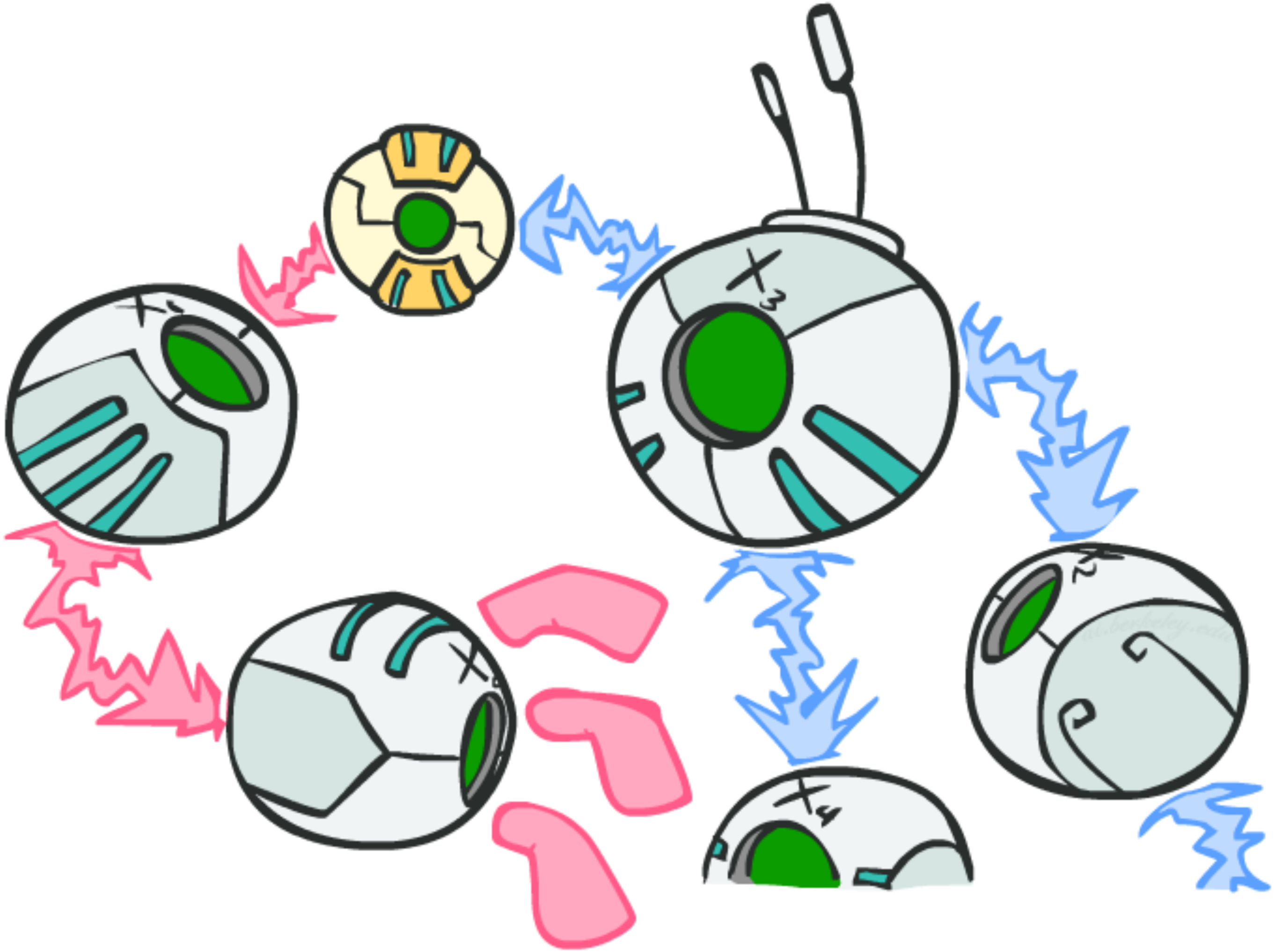
Independence in a BN

- Important question about a BN:
  - Are two nodes independent given certain evidence?
  - If yes, can prove using algebra (tedious in general)
  - If no, can prove with a counter example
  - Example:



  - Question: are X and Z necessarily independent?
    - Answer: no. Example: low pressure causes rain, which causes traffic.
    - X can influence Z, Z can influence X (via Y)
    - Addendum: they *could* be independent: how?

D-separation: Outline



D-separation: Outline

- Study independence properties for triples
- Analyze complex cases in terms of member triples
- D-separation: a condition / algorithm for answering such queries

Causal Chains

- This configuration is a “causal chain”

X: Low pressure　　Y: Rain　　Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z ? *No!*
  - One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
  - Example:
    - Low pressure causes rain causes traffic, high pressure causes no rain causes no traffic
    - In numbers:

      P(+y | +x) = 1, P(-y | - x) = 1,
      P(+z | +y) = 1, P(-z | -y) = 1

Causal Chains

- This configuration is a "causal chain"

X: Low pressure Y: Rain Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

$$= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)}$$

$$= P(z|y)$$

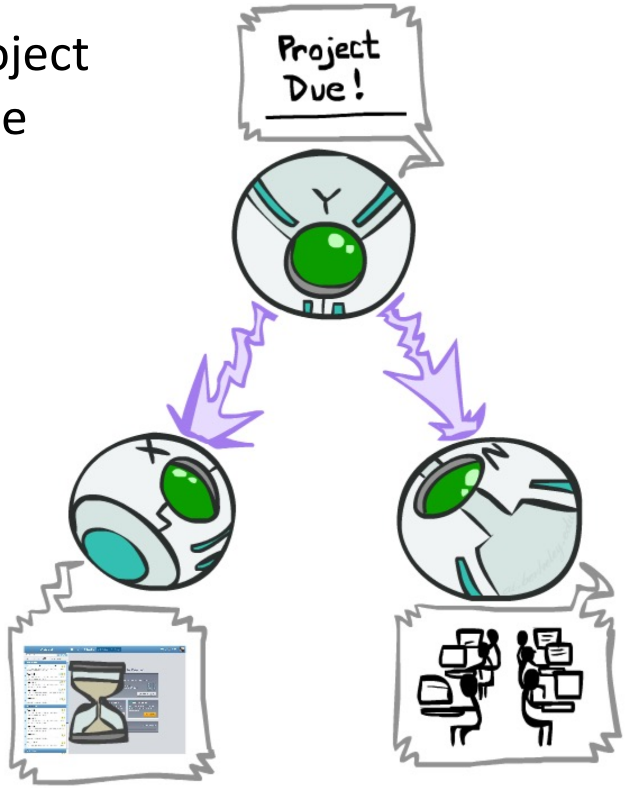
Yes!

- Evidence along the chain "blocks" the influence

Common Cause

- This configuration is a "common cause"

Y: Project due



X: Forums busy

Z: Lab full

$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

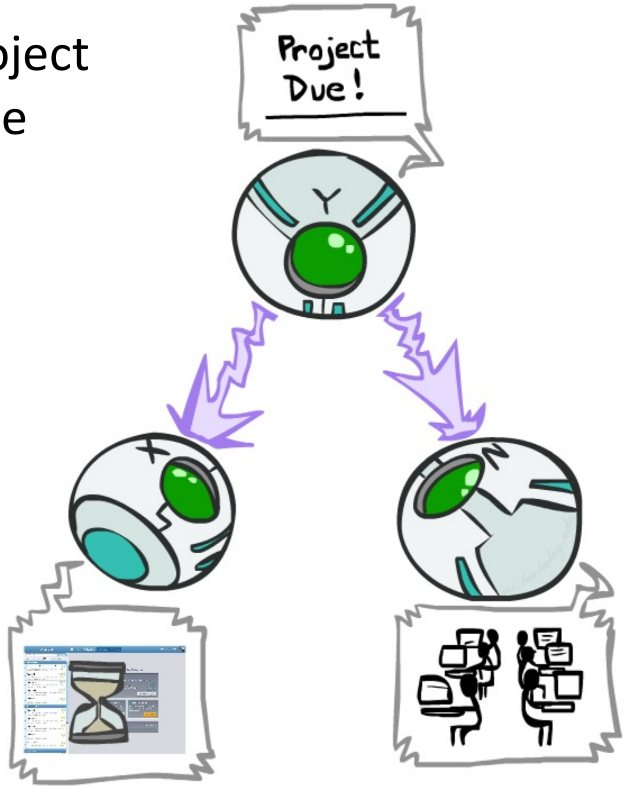
- Guaranteed X independent of Z ? *No!*
  - One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
  - Example:
    - Project due causes both forums busy and lab full
    - In numbers:

      P(+x | +y) = 1, P(-x | -y) = 1,
      P(+z | +y) = 1, P(-z | -y) = 1

Common Cause

- This configuration is a "common cause"

Y: Project due



X: Forums busy

Z: Lab full

$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X and Z independent given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

$$= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)}$$

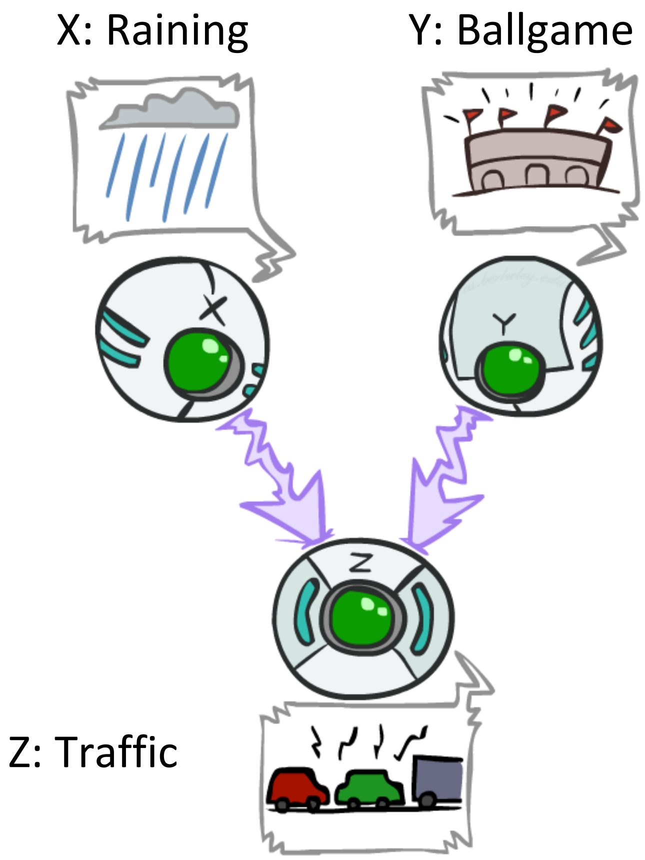
$$= P(z|y)$$

Yes!

- Observing the cause blocks influence between effects.

Common Effect

- Last configuration: two causes of one effect (v-structures)



- Are X and Y independent?
  - *Yes*: the ballgame and the rain cause traffic, but they are not correlated
  - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
  - *No*: seeing traffic puts the rain and the ballgame in competition as explanation.
- This is backwards from the other cases

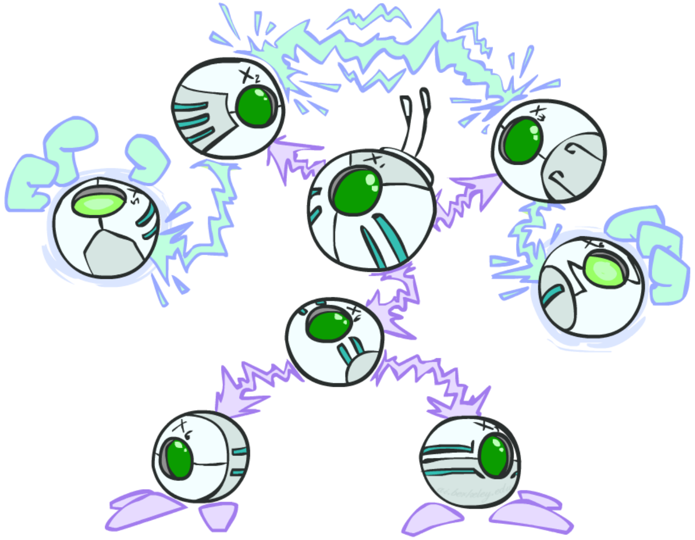
The General Case

The General Case

- General question: in a given BN, are two variables independent (given evidence)?

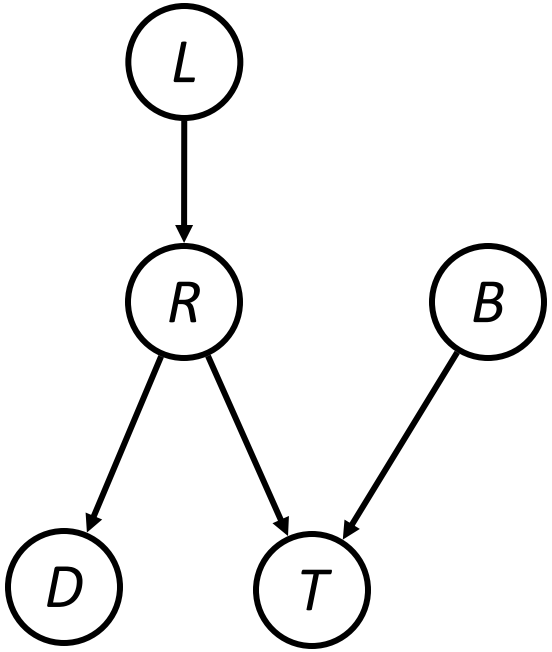
- Solution: analyze the graph

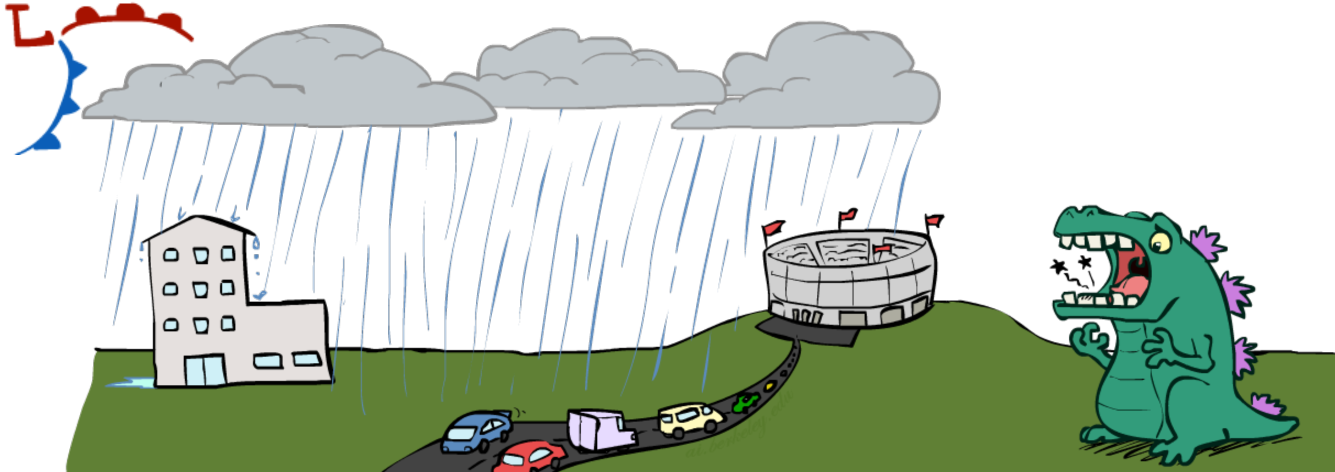
- Any complex example can be broken
  into repetitions of the three canonical cases



Reachability

- Recipe: shade evidence nodes, look for paths in the resulting graph
- Attempt 1: if two nodes are connected by an undirected path not blocked by a shaded observed node, they are not conditionally independent
  - Influence can "flow" between them, unblocked
- Almost works, but not quite
  - Where does it break?
  - Answer: the v-structure at T doesn't count as a link in a path unless "active" via being observed as evidence



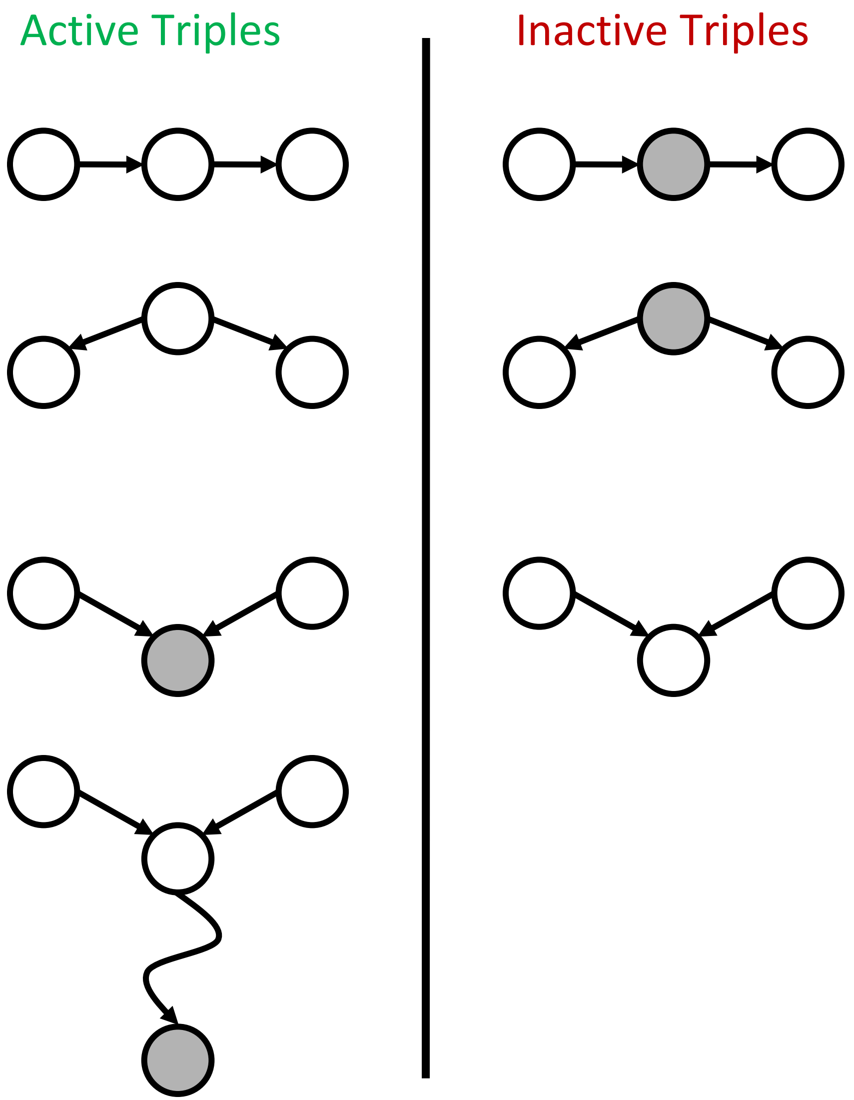


Active / Inactive Paths

- Question: Are X and Y conditionally independent given evidence variables {Z}?
  - Yes, if X and Y "d-separated" by Z
  - Consider all (undirected) paths from X to Y
  - No active paths = conditional independence!

- A path is active if each triple is active:
  - Causal chain A → B → C where B is unobserved (either direction)
  - Common cause A ← B → C where B is unobserved
  - Common effect (aka v-structure)
    A → B ← C where B *or one of its descendants* is observed

- All it takes to block a path is a single inactive segment



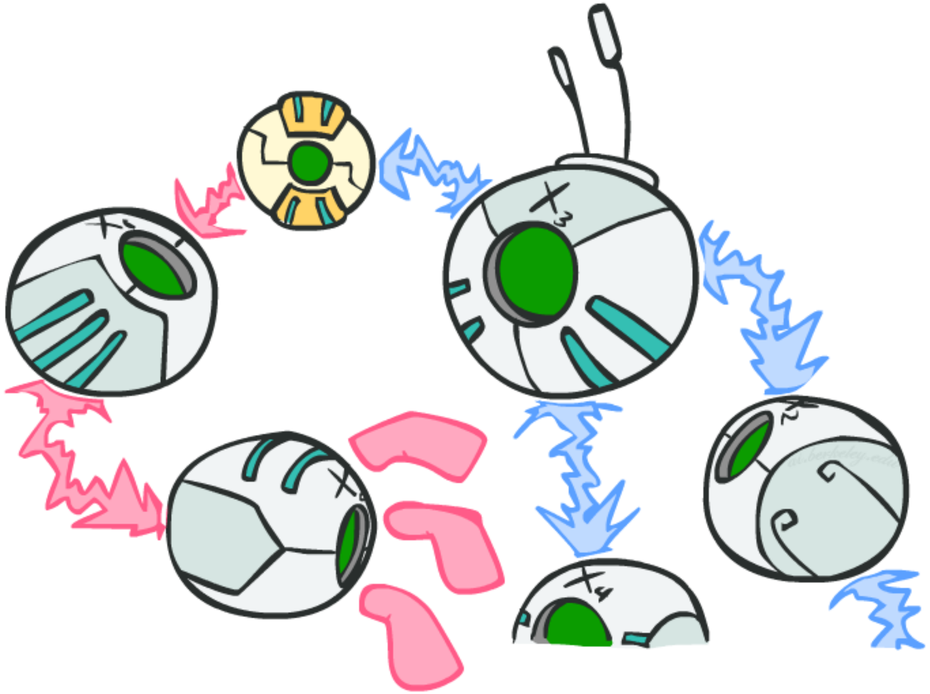
D-Separation

- Query: $X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$?
- Check all (undirected!) paths between X_i and X_j
  - If one or more active, then independence not guaranteed

$$X_i \not\perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

  - Otherwise (i.e. if all paths are inactive), then independence is guaranteed

$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

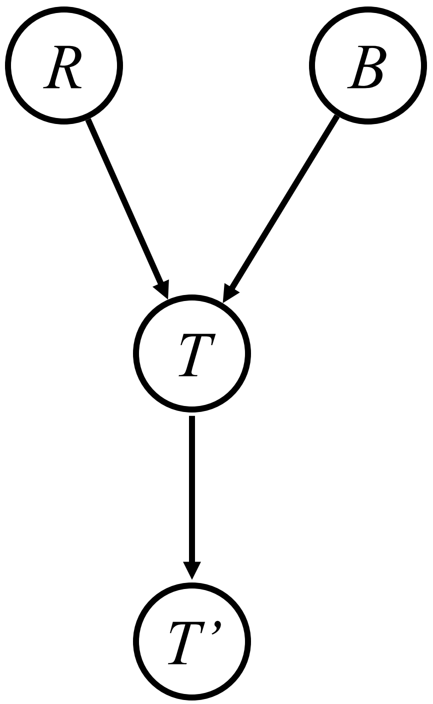


Example

$R \perp\!\!\!\perp B$ *Yes*

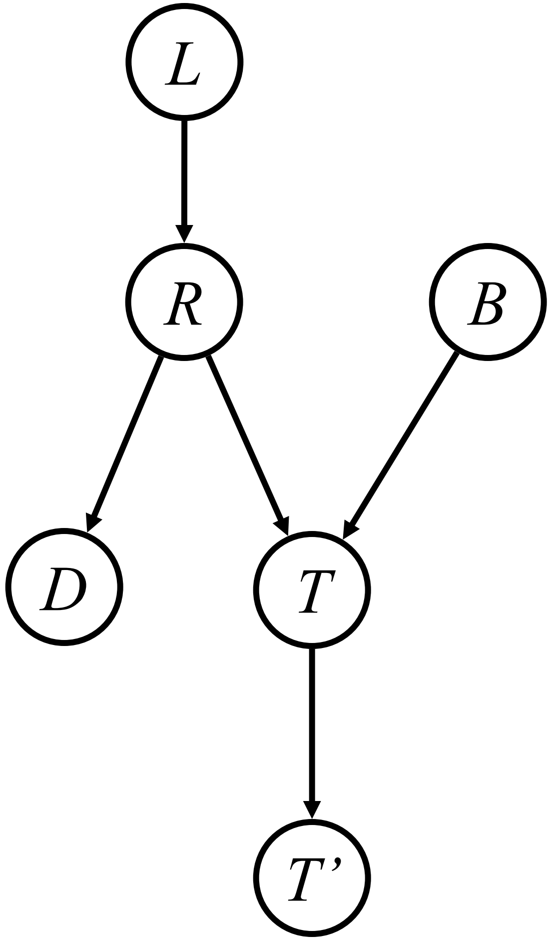
$R \perp\!\!\!\perp B | T$ *Not guaranteed*

$R \perp\!\!\!\perp B | T'$ *Not guaranteed*



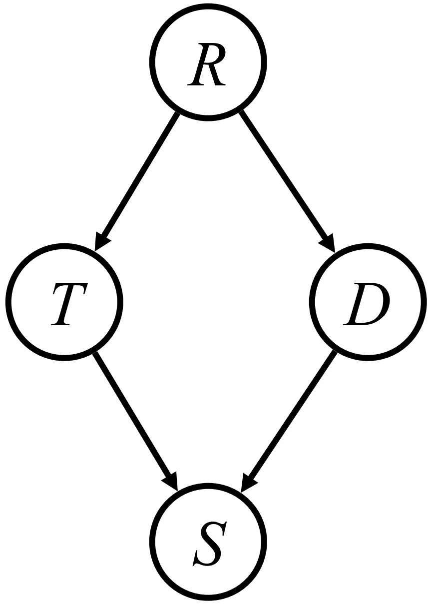
Example

$L \perp\!\!\!\perp T' \mid T$	*Yes*
$L \perp\!\!\!\perp B$	*Yes*
$L \perp\!\!\!\perp B \mid T$	*Not guaranteed*
$L \perp\!\!\!\perp B \mid T'$	*Not guaranteed*
$L \perp\!\!\!\perp B \mid T, R$	*Yes*



Example

- Variables:
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I’m sad
- Questions:



$T \perp\!\!\!\perp D$ *Not guaranteed*

$T \perp\!\!\!\perp D | R$ *Yes*

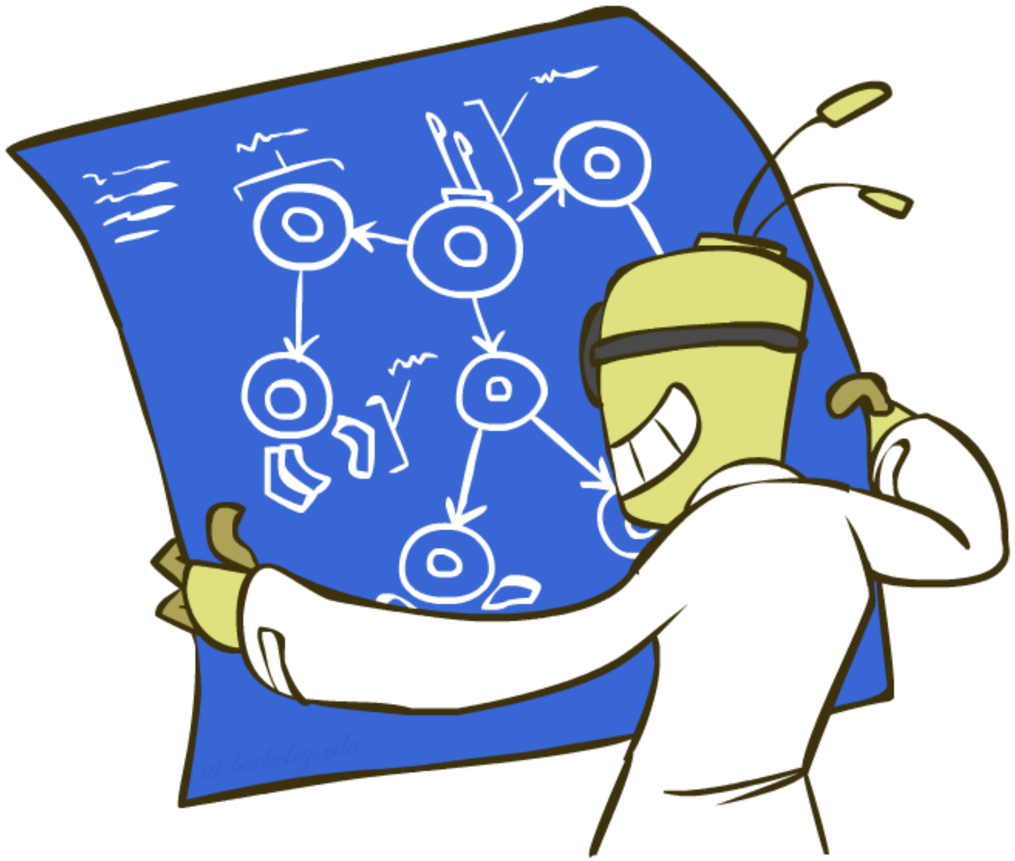
$T \perp\!\!\!\perp D | R, S$ *Not guaranteed*

Structure Implications

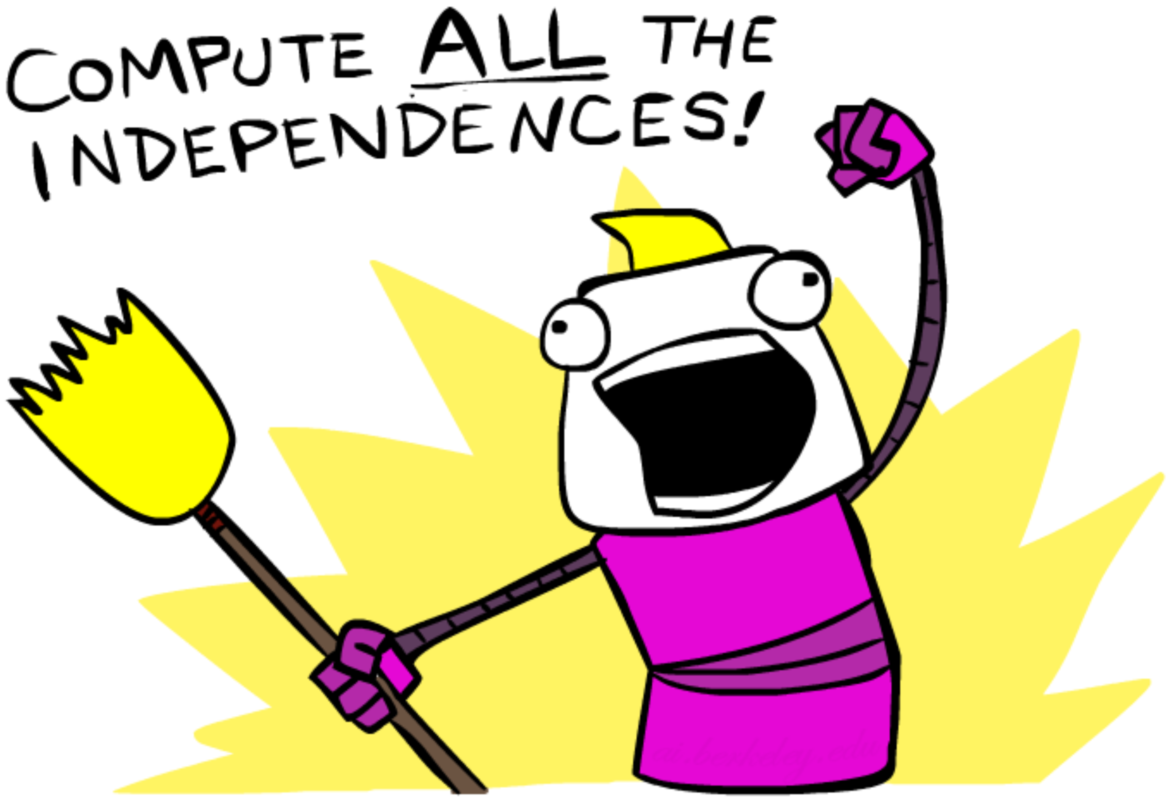
- Given a Bayes net structure, can run d-separation algorithm to build a complete list of conditional independences that are necessarily true of the form

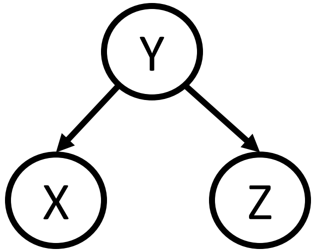
$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

- This list determines the set of probability distributions that can be represented

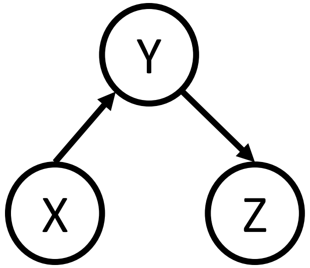


Computing All Independences

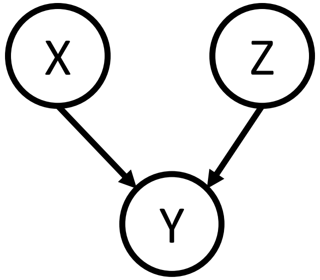




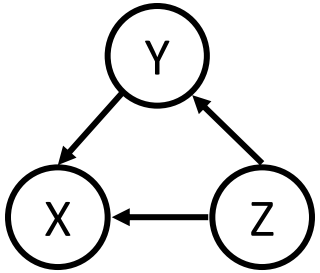
$$X \perp\!\!\!\perp Z \mid Y$$



$$X \perp\!\!\!\perp Z \mid Y$$



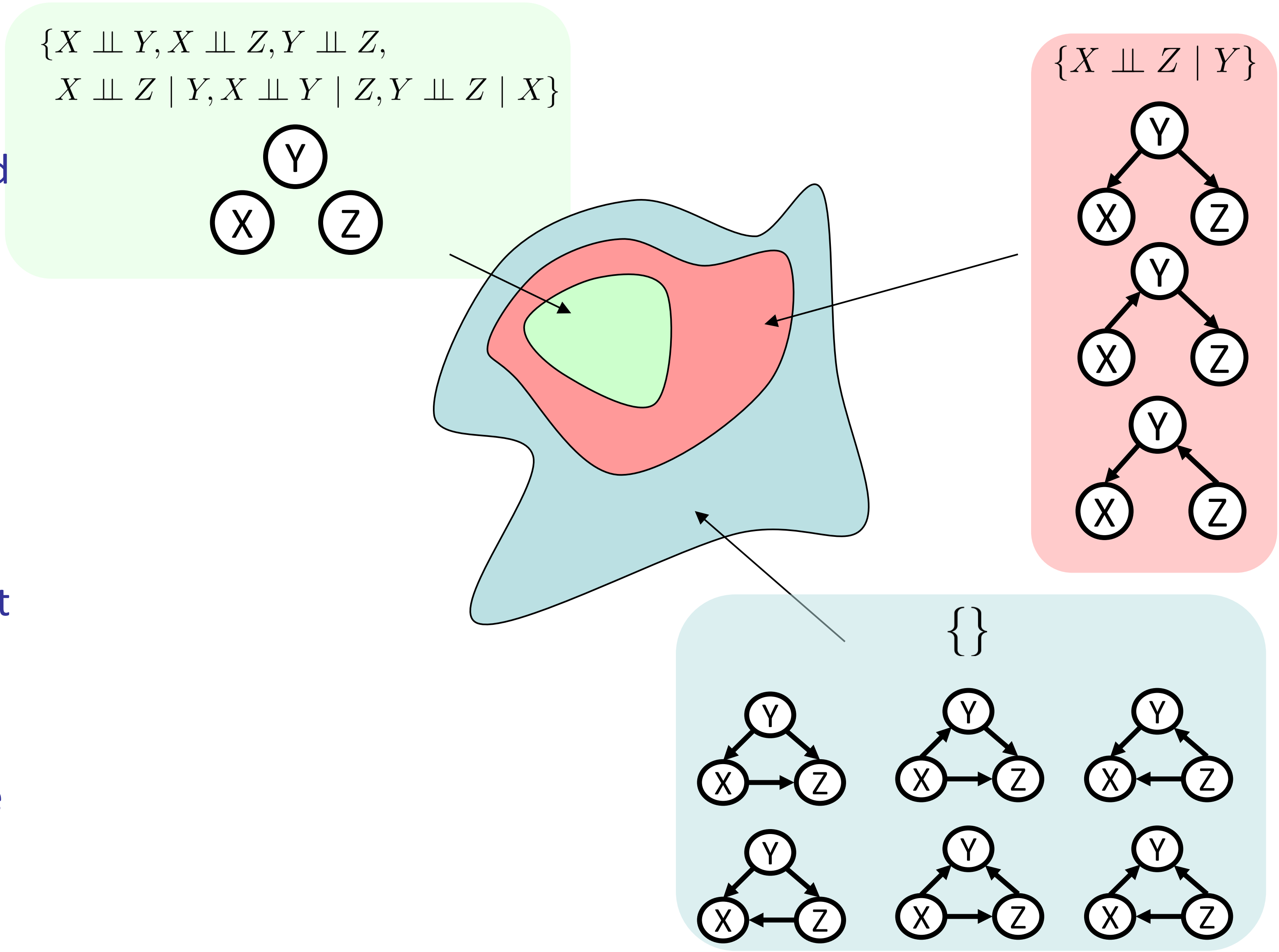
$$X \perp\!\!\!\perp Z$$



None!

Topology Limits Distributions

- Given some graph topology G, only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution



Bayes Nets Representation Summary

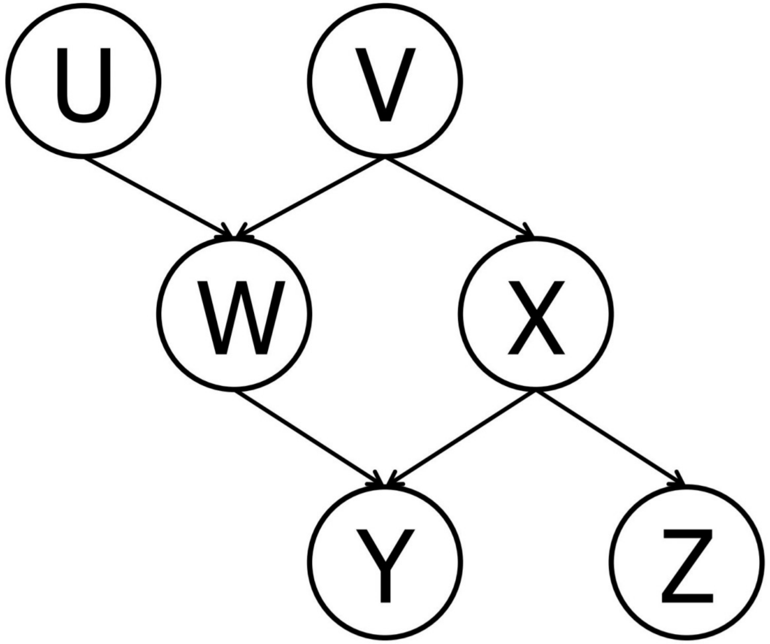
- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

Exercises

Independence

Based only on the structure of the Bayes' Net given right, indicate whether the following conditional independence assertions are either 1) *guaranteed to be true*, 2) *guaranteed to be false*, or 3) *cannot be determined* by the structure alone.
Hint:

- the meaning of $A \perp B \mid C, D$ is A and B are independent ($\perp$) of each other conditioned on (|) C and D
- Two properties about independence are in Fig 14.4 (Page 518) in the textbook.



i. $U \perp V$
ii. $U \perp V \mid W$
iii. $U \perp Z \mid W$
iv. $U \perp Z \mid X, W$
v. $V \perp Z \mid X$

Bayes Nets

- ✓ Representation
- ✓ Conditional Independences
- Probabilistic Inference
  - Enumeration (exact, exponential complexity)
  - Variable elimination (exact, worst-case exponential complexity, often better)
  - Probabilistic inference is NP-complete
  - Sampling (approximate)
- Learning Bayes' Nets from Data