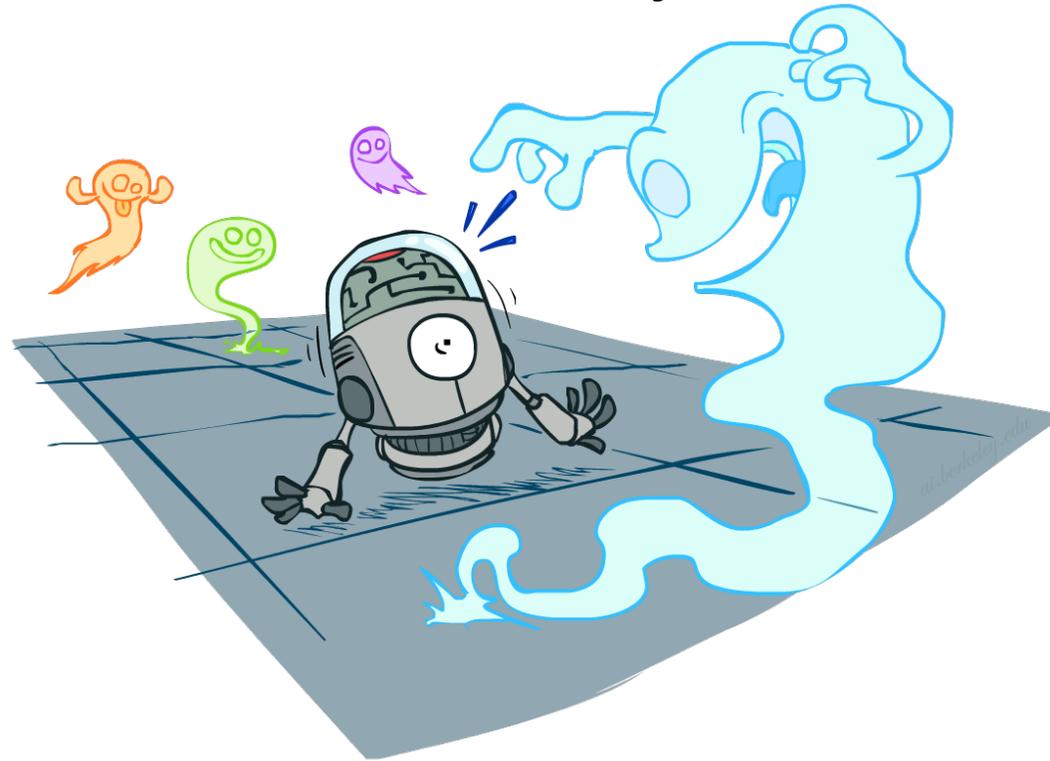


CS 343: Artificial Intelligence

Probability



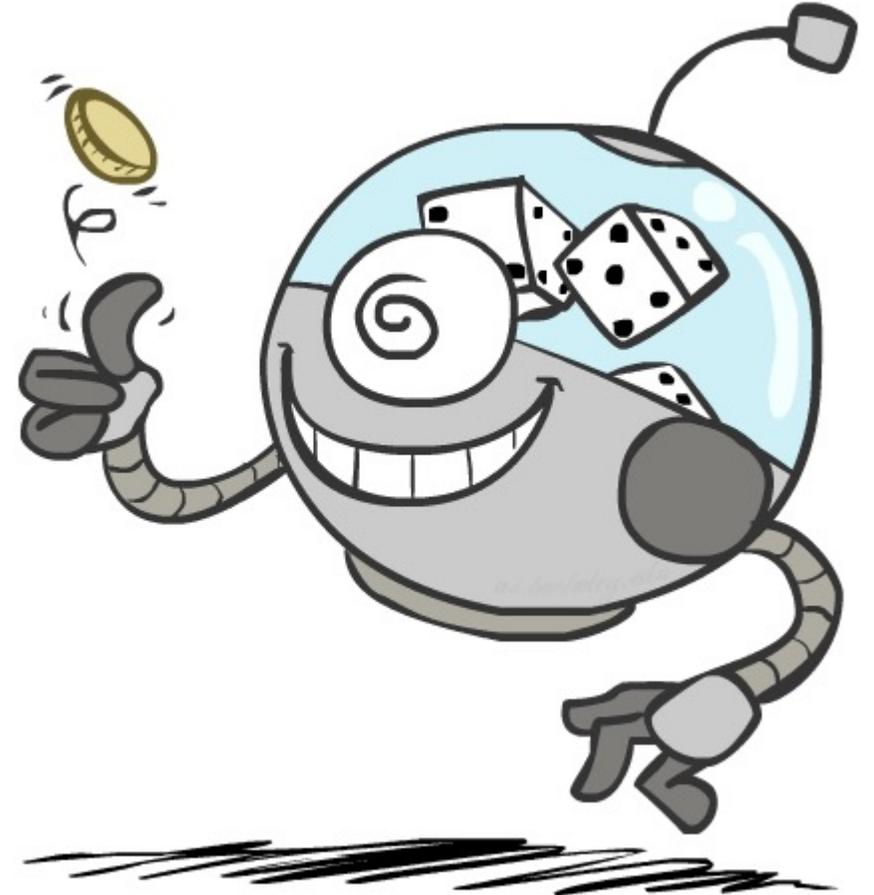
Prof. Yuke Zhu — The University of Texas at Austin

Announcements

- The midterm will be released at 8:00 am on March 10, Thursday.
- The midterm is based on the content of weeks 2 - 7.
- There is no class on March 10.
- The expected time is 90 minutes + 15 minutes for uploading answers (so 105 minutes total from the time the exam begins).
- The midterm must be completed by midnight on Friday (so started by 10:15 pm on Friday).
- The exam will be held on Gradescope.

Today

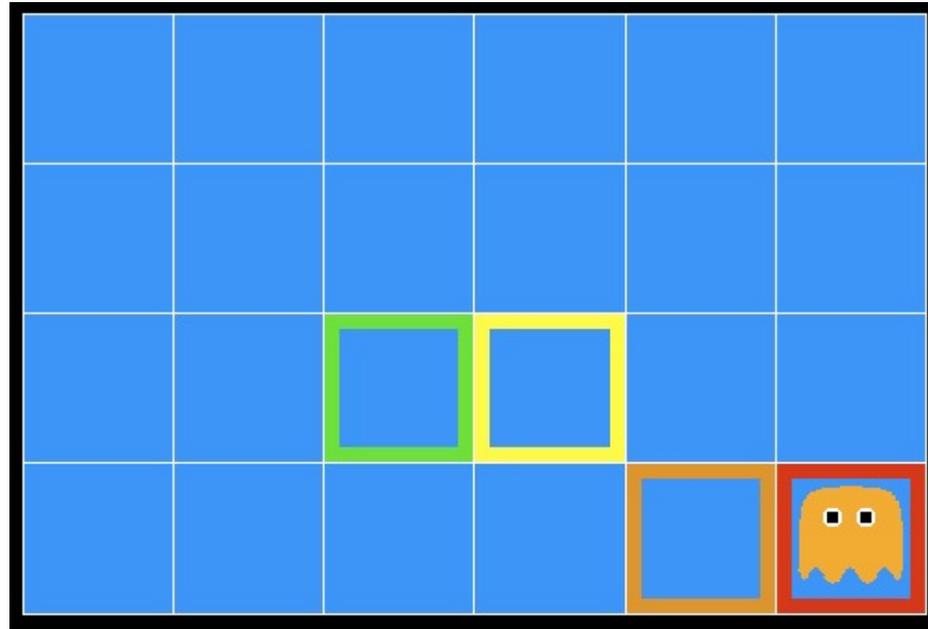
- Probability
 - Random Variables
 - Joint and Marginal Distributions
 - Conditional Distributions
 - Product Rule, Chain Rule, Bayes' Rule
 - Inference
 - Independence
- You'll need all this stuff A LOT for the next few weeks, so make sure you go over it now!



Inference in Ghostbusters

- A ghost is in the grid somewhere
- Noisy sensor readings tell how close a square is to the ghost.
Most likely observations:

- On the ghost: red
- 1 or 2 away: orange
- 3 or 4 away: yellow
- 5+ away: green



- Sensors are noisy, but we know $P(\text{Color} \mid \text{Distance})$

$P(\text{red} \mid 3)$	$P(\text{orange} \mid 3)$	$P(\text{yellow} \mid 3)$	$P(\text{green} \mid 3)$
0.05	0.15	0.5	0.3

Ghostbusters, no probabilities

The image shows a screenshot of a web-based game titled "ghostbusters" with the subtitle "Inference in Ghostbusters". The main interface consists of a 10x10 grid of blue squares. To the right of the grid, there is a panel with the following text:

GHOSTS REMAINING: 1
BUSTS REMAINING: 1
SCORE: 0

MESSAGES:

BUST

TIME+1

Below the grid, there are two buttons labeled "0.05" and "0.15". At the bottom of the interface, there is a terminal window with the following text:

```
Enter any index to play any demo and up to go to the upper menu  
1  
Here are the instructions about how to run it: Click the grid to guess and try t  
o bust the ghost
```

On the right side of the terminal window, there is a scroll bar and some partially visible text: "7, 18, 19, 2".

Uncertainty

- General situation:
 - **Observed variables (evidence):** Agent knows certain things about the state of the world (e.g., sensor readings or symptoms)
 - **Unobserved variables:** Agent needs to reason about other aspects (e.g. where an object is or what disease is present)
 - **Model:** Agent knows something about how the known variables relate to the unknown variables
- Probabilistic reasoning gives us a framework for using beliefs and knowledge to perform *inference*

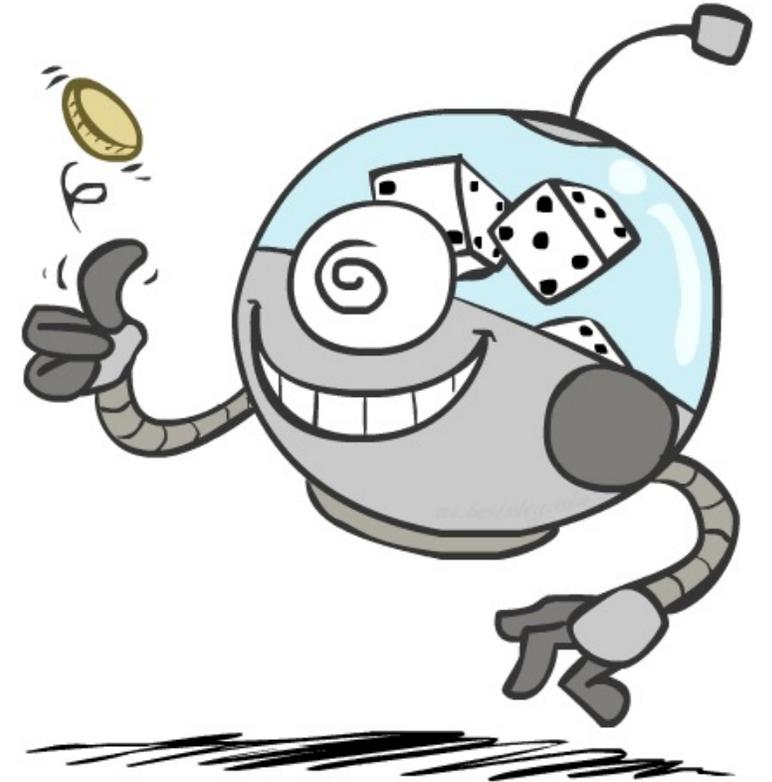
0.11	0.11	0.11
0.11	0.11	0.11
0.11	0.11	0.11

0.17	0.10	0.10
0.09	0.17	0.10
<0.01	0.09	0.17

<0.01	<0.01	0.03
<0.01	0.05	0.05
<0.01	0.05	0.81

Random Variables

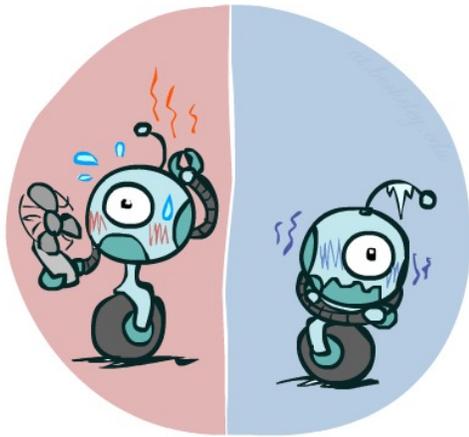
- A random variable is some aspect of the world about which we (may) have uncertainty
 - R = Is it raining?
 - T = Is it hot or cold?
 - D = How long will it take to drive to work?
 - L = Where is the ghost?
- We denote random variables with capital letters
- Like variables in a CSP, random variables have domains
 - R in {true, false} (often write as {+r, -r})
 - T in {hot, cold}
 - D in $[0, \infty)$
 - L in possible locations, maybe $\{(0,0), (0,1), \dots\}$



Probability Distributions

- Associate a probability with each value

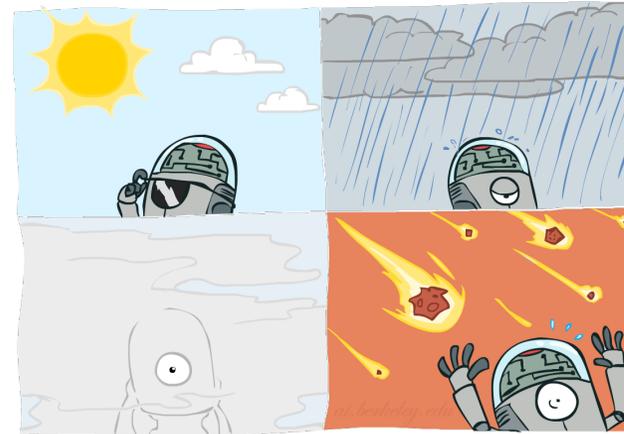
- Temperature:



$P(T)$

T	P
hot	0.5
cold	0.5

- Weather:



$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Probability Distributions

- Unobserved random variables have distributions

$$P(T)$$

T	P
hot	0.5
cold	0.5

$$P(W)$$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Shorthand notation:

$$\begin{aligned}P(\textit{hot}) &= P(T = \textit{hot}), \\P(\textit{cold}) &= P(T = \textit{cold}), \\P(\textit{rain}) &= P(W = \textit{rain}), \\&\dots\end{aligned}$$

- A discrete distribution is a table of probabilities of values
- A probability (lower case value) is a single number

OK if all domain entries are unique

$$P(W = \textit{rain}) = 0.1$$

- Must have: $\forall x \ P(X = x) \geq 0$ and $\sum_x P(X = x) = 1$

Joint Distributions

- A *joint distribution* over a set of random variables: X_1, X_2, \dots, X_n specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$

$$P(x_1, x_2, \dots, x_n)$$

- Must obey:

$$P(x_1, x_2, \dots, x_n) \geq 0$$

$$\sum_{(x_1, x_2, \dots, x_n)} P(x_1, x_2, \dots, x_n) = 1$$

- Size of distribution if n variables with domain sizes d ?
 - For all but the smallest distributions, impractical to write out!

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Probabilistic Models

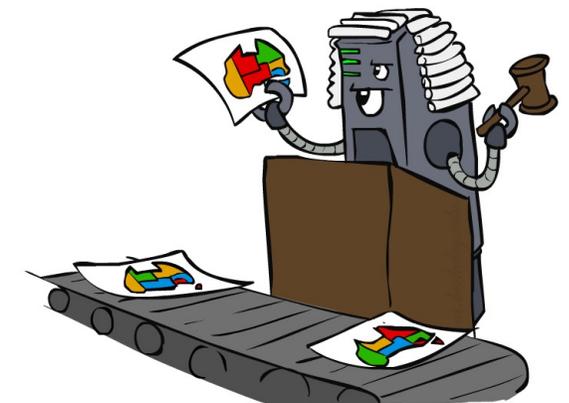
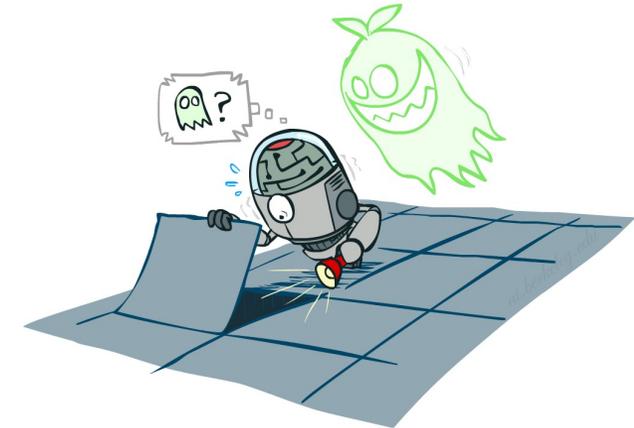
- A probabilistic model is a joint distribution over a set of random variables
- Probabilistic models:
 - (Random) variables with domains
 - Assignments are called *outcomes*
 - Joint distributions: say whether assignments (outcomes) are likely
 - *Normalized*: sum to 1.0
 - Ideally: only certain variables directly interact
- Constraint satisfaction problems:
 - Variables with domains
 - Constraints: state whether assignments are possible
 - Ideally: only certain variables directly interact

Distribution over T,W

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Constraint over T,W

T	W	P
hot	sun	T
hot	rain	F
cold	sun	F
cold	rain	T



Events

- An *event* is a set E of outcomes

$$P(E) = \sum_{(x_1 \dots x_n) \in E} P(x_1 \dots x_n)$$

- From a joint distribution, we can calculate the probability of any event
 - Probability that it's hot AND sunny?
 - Probability that it's hot?
 - Probability that it's hot OR sunny?
- Typically, the events we care about are *partial assignments*, like $P(T=\text{hot})$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Quiz: Events

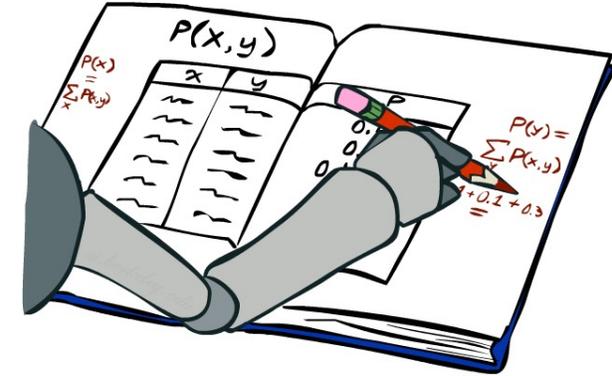
- $P(+x, +y)$?
- $P(+x)$?
- $P(-y \text{ OR } +x)$?

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Marginal Distributions

- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding



$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3



$$P(t) = \sum_s P(t, s)$$

$P(T)$

T	P
hot	0.5
cold	0.5



$$P(s) = \sum_t P(t, s)$$

$P(W)$

W	P
sun	0.6
rain	0.4

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

Quiz: Marginal Distributions

$P(X, Y)$

X	Y	P
+x	+y	0.1
+x	-y	0.5
-x	+y	0.2
-x	-y	0.2



$$P(x) = \sum_y P(x, y)$$



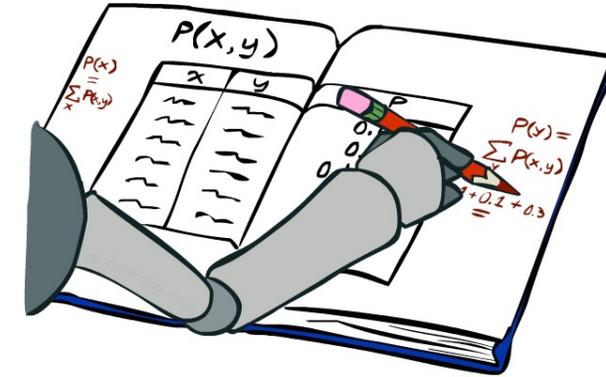
$$P(y) = \sum_x P(x, y)$$

$P(X)$

X	P
+x	
-x	

$P(Y)$

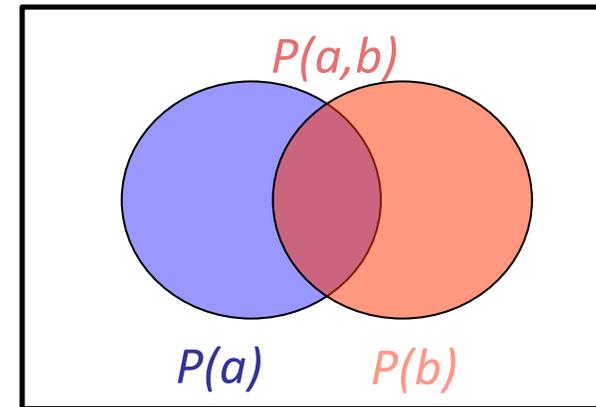
Y	P
+y	
-y	



Conditional Probabilities

- A simple relation between joint and conditional probabilities
 - In fact, this is taken as the *definition* of a conditional probability

$$P(a|b) = \frac{P(a, b)}{P(b)}$$



$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W = s|T = c) = \frac{P(W = s, T = c)}{P(T = c)} = \frac{0.2}{0.5} = 0.4$$

$$\begin{aligned} &= P(W = s, T = c) + P(W = r, T = c) \\ &= 0.2 + 0.3 = 0.5 \end{aligned}$$

Quiz: Conditional Probabilities

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- $P(+x \mid +y)$?

- $P(-x \mid +y)$?

- $P(-y \mid +x)$?

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

$P(W|T)$

$P(W T = hot)$	
W	P
sun	0.8
rain	0.2

$P(W T = cold)$	
W	P
sun	0.4
rain	0.6

Joint Distribution

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$\begin{aligned}P(W = s|T = c) &= \frac{P(W = s, T = c)}{P(T = c)} \\ &= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\ &= \frac{0.2}{0.2 + 0.3} = 0.4\end{aligned}$$



$$\begin{aligned}P(W = r|T = c) &= \frac{P(W = r, T = c)}{P(T = c)} \\ &= \frac{P(W = r, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\ &= \frac{0.3}{0.2 + 0.3} = 0.6\end{aligned}$$

$P(W|T = c)$

W	P
sun	0.4
rain	0.6

Normalization Trick

$$\begin{aligned}
 P(W = s|T = c) &= \frac{P(W = s, T = c)}{P(T = c)} \\
 &= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\
 &= \frac{0.2}{0.2 + 0.3} = 0.4
 \end{aligned}$$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence



$P(c, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one)



$P(W|T = c)$

W	P
sun	0.4
rain	0.6

$$\begin{aligned}
 P(W = r|T = c) &= \frac{P(W = r, T = c)}{P(T = c)} \\
 &= \frac{P(W = r, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\
 &= \frac{0.3}{0.2 + 0.3} = 0.6
 \end{aligned}$$

Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence



$P(c, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one)



$P(W|T = c)$

W	P
sun	0.4
rain	0.6

• Why does this work?

$$P(x_1|x_2) = \frac{P(x_1, x_2)}{P(x_2)} = \frac{P(x_1, x_2)}{\sum_{x_1} P(x_1, x_2)}$$

Quiz: Normalization Trick

- $P(X | Y=-y)$?

$P(X, Y)$

X	Y	P
+x	+y	0.3
+x	-y	0.1
-x	+y	0.5
-x	-y	0.1

SELECT the joint probabilities matching the evidence



NORMALIZE the selection (make it sum to one)



To Normalize

- (Dictionary) To bring or restore to a normal condition

All entries sum to ONE

- Procedure:

- Step 1: Compute $Z = \text{sum over all entries}$
- Step 2: Divide every entry by Z

- Example 1

W	P
sun	0.2
rain	0.3

Normalize
Z = 0.5

W	P
sun	0.4
rain	0.6

- Example 2

T	W	P
hot	sun	20
hot	rain	5
cold	sun	10
cold	rain	15

Normalize
Z = 50

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Probabilistic Inference

- Probabilistic inference: compute a desired probability from other known probabilities (e.g. conditional from joint)
- We generally compute conditional probabilities
 - $P(\text{on time} \mid \text{no reported accidents}) = 0.90$
 - These represent the agent's *beliefs* given the evidence
- Probabilities change with new evidence:
 - $P(\text{on time} \mid \text{no accidents, 5 a.m.}) = 0.95$
 - $P(\text{on time} \mid \text{no accidents, 5 a.m., raining}) = 0.80$
 - Observing new evidence causes *beliefs to be updated*



Inference by Enumeration

- General case:

- Evidence variables: $E_1 \dots E_k = e_1 \dots e_k$
 - Query* variable: Q
 - Hidden variables: $H_1 \dots H_r$
- } X_1, X_2, \dots, X_n
All variables

- We want:

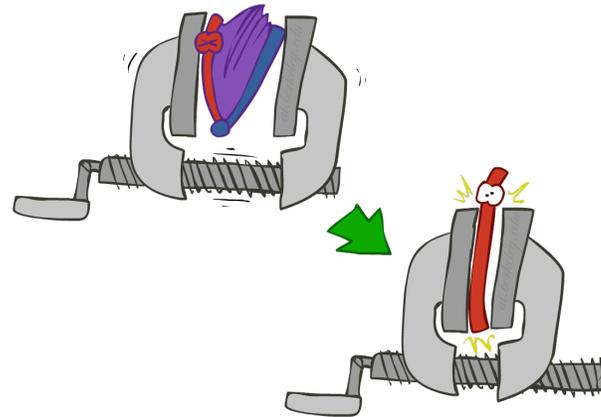
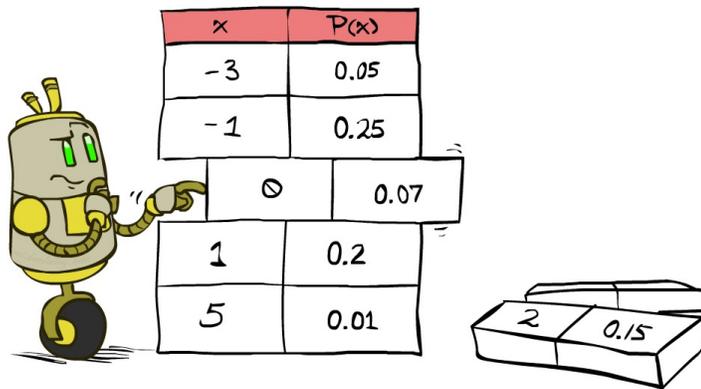
$$P(Q|e_1 \dots e_k)$$

** Works fine with multiple query variables, too*

- Step 1: Select the entries consistent with the evidence

- Step 2: Sum out H to get joint of Query and evidence

- Step 3: Normalize



$$\times \frac{1}{Z}$$

$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} P(Q, \underbrace{h_1 \dots h_r}_{X_1, X_2, \dots, X_n}, e_1 \dots e_k)$$

$$Z = \sum_q P(Q, e_1 \dots e_k)$$

$$P(Q|e_1 \dots e_k) = \frac{1}{Z} P(Q, e_1 \dots e_k)$$

Inference by Enumeration

- $P(W)$?

$$p(W=\text{sun}) = 0.3 + 0.1 + 0.1 + 0.15 = 0.65$$

$$p(W=\text{rain}) = 0.05 + 0.05 + 0.05 + 0.2 = 0.35$$

- $P(W \mid \text{winter})$?

$$p(W=\text{sun} , \text{winter}) = 0.1 + 0.15 = 0.25$$

$$p(W=\text{rain} , \text{winter}) = 0.05 + 0.2 = 0.25$$

$$p(W=\text{sun} \mid \text{winter}) = 0.25 / (0.25 + 0.25) = 0.5$$

$$p(W=\text{rain} \mid \text{winter}) = 0.25 / (0.25 + 0.25) = 0.5$$

- $P(W \mid \text{winter}, \text{hot})$?

$$p(W=\text{sun} , \text{winter}, \text{hot}) = 0.1$$

$$p(W=\text{rain} , \text{winter}, \text{hot}) = 0.05$$

$$p(W=\text{sun} \mid \text{winter}, \text{hot}) = 0.1 / (0.1 + 0.05) = 2/3$$

$$p(W=\text{rain} \mid \text{winter}, \text{hot}) = 0.05 / (0.1 + 0.05) = 1/3$$

S	T	W	P
summer	hot	sun	0.30
summer	hot	rain	0.05
summer	cold	sun	0.10
summer	cold	rain	0.05
winter	hot	sun	0.10
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.20

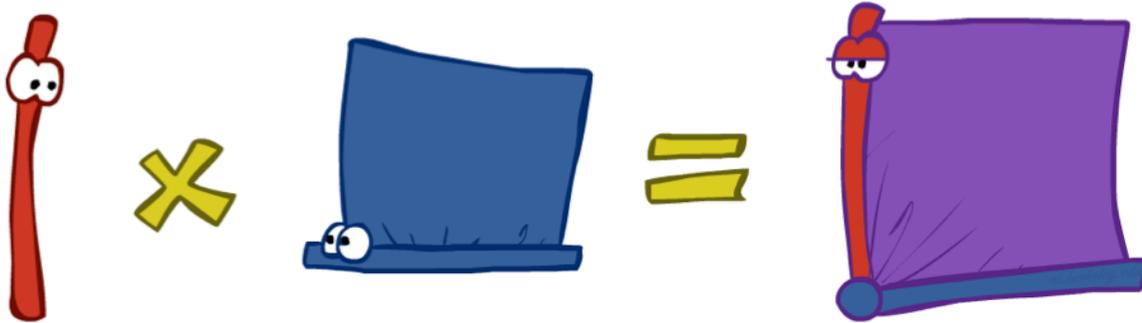
Inference by Enumeration

- Obvious problems:
 - Worst-case time complexity $O(d^n)$
 - Space complexity $O(d^n)$ to store the joint distribution
 - What about continuous distributions?

The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(y)P(x|y) = P(x, y) \quad \longleftrightarrow \quad P(x|y) = \frac{P(x, y)}{P(y)}$$



The Product Rule

$$P(y)P(x|y) = P(x, y)$$

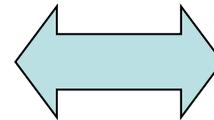
- Example:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3



$P(D, W)$

D	W	P
wet	sun	
dry	sun	
wet	rain	
dry	rain	

The Chain Rule

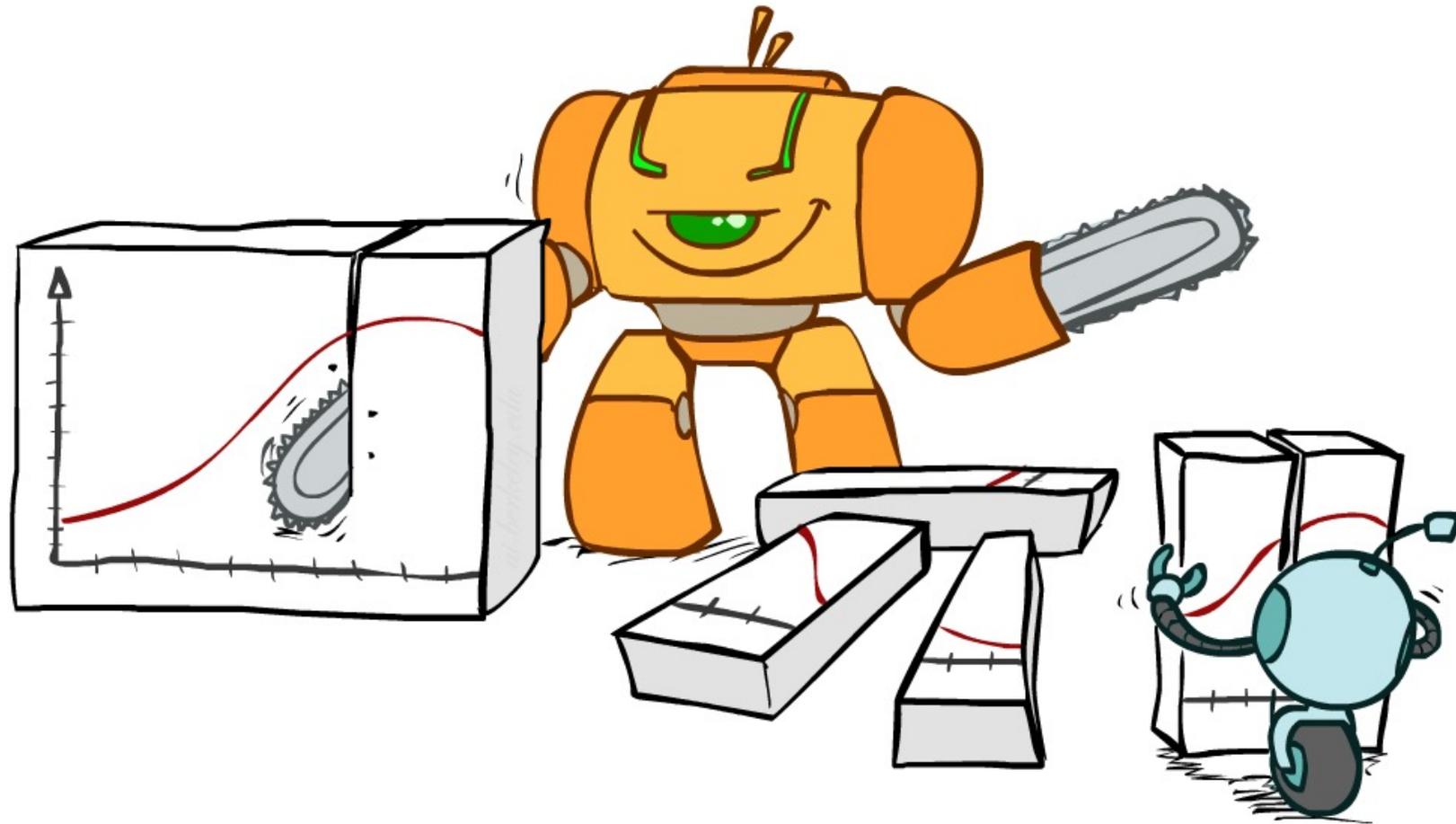
- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \dots, x_n) = \prod_i P(x_i|x_1 \dots x_{i-1})$$

- Why is this always true?

Bayes' Rule



Bayes' Rule

- Two ways to factor a joint distribution over two variables:

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)}$$

likelihood

prior

normalization

- Why is this at all helpful?
 - Lets us build one conditional from its reverse
 - Often one conditional is tricky but the other one is simple
 - Foundation of many systems we'll see later
- In the running for most important AI equation!

That's my rule!



Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

- Example:

- M: meningitis, S: stiff neck

$$\left. \begin{aligned} P(+m) &= 0.0001 \\ P(+s|+m) &= 0.8 \\ P(+s|-m) &= 0.01 \end{aligned} \right\} \text{Example givens}$$

$$P(+m|+s) = \frac{P(+s|+m)P(+m)}{P(+s)} = \frac{P(+s|+m)P(+m)}{P(+s|+m)P(+m) + P(+s|-m)P(-m)} = \frac{0.8 \times 0.0001}{0.8 \times 0.0001 + 0.01 \times 0.999}$$

- Note: posterior probability of meningitis still very small
- Note: you should still get stiff necks checked out! Why?

=0.0008

Quiz: Bayes' Rule

- Given:

$$P(W)$$

R	P
sun	0.8
rain	0.2

$$P(D|W)$$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- What is $P(W | \text{dry})$?

$$p(\text{sun} | \text{dry}) = p(\text{dry} | \text{sun}) p(\text{sun}) / p(\text{dry}) = 0.9 * 0.8 / Z = .72 / Z$$

$$p(\text{rain} | \text{dry}) = p(\text{dry} | \text{rain}) p(\text{rain}) / p(\text{dry}) = 0.3 * 0.2 / Z = 0.06 / Z$$

$$Z = .72 + .06 = .78$$

Ghostbusters, Revisited

- Let's say we have two distributions:
 - Prior distribution** over ghost location: $P(G)$
 - Let's say this is uniform
 - Sensor reading model: $P(R | G)$
 - Given: we know what our sensors do
 - R = reading color measured at $(1,1)$
 - E.g. $P(R = \text{yellow} | G=(1,1)) = 0.1$
- We can calculate the **posterior distribution** $P(G|r)$ over ghost locations given a reading using Bayes' rule:

$$P(g|r) \propto P(r|g)P(g)$$

0.11	0.11	0.11
0.11	0.11	0.11
0.11	0.11	0.11

0.17	0.10	0.10
0.09	0.17	0.10
<0.01	0.09	0.17

Ghostbusters with Probability

Ghostbusters, Revisited

The screenshot shows the 'ghostbusters' game window. The main area is a 10x10 grid of blue squares, each containing a probability value. The values are: 0.04, 0.04, 0.04, <0.01, <0.01, <0.01, 0.04, 0.04, 0.04, 0.04 in the first row; 0.04, 0.04, <0.01, <0.01, <0.01, <0.01, <0.01, 0.04, 0.04, 0.04 in the second row; 0.04, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, 0.04, 0.04 in the third row; <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, 0.04 in the fourth row; 0.04, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, <0.01, 0.04, 0.04 in the fifth row; 0.04, 0.04, <0.01, <0.01, <0.01, <0.01, <0.01, 0.04, 0.04, 0.04 in the sixth row. The square at row 4, column 5 is highlighted with a green border. To the right of the grid, the game statistics are displayed: GHOSTS REMAINING: 1, BUSTS REMAINING: 1, SCORE: -1. Below the statistics, the messages section shows: MESSAGES: sensor at (3, 4) [GREEN]. At the bottom right, there are two buttons: a blue 'BUST' button and a grey 'TIME+1' button. A terminal window is partially visible on the right side of the game window, showing the text '18, 19, 2' and 'and try t'.

0.04	0.04	0.04	<0.01	<0.01	<0.01	0.04	0.04	0.04	0.04
0.04	0.04	<0.01	<0.01	<0.01	<0.01	<0.01	0.04	0.04	0.04
0.04	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.04	0.04
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.04
0.04	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.04	0.04
0.04	0.04	<0.01	<0.01	<0.01	<0.01	<0.01	0.04	0.04	0.04

GHOSTS REMAINING: 1
BUSTS REMAINING: 1
SCORE: -1

MESSAGES:
sensor at (3, 4) [GREEN]

BUST

TIME+1

18, 19, 2
and try t