Uncertain Outcomes
Idea: Uncertain outcomes controlled by chance, not an adversary!
Expectimax Search

- Why wouldn’t we know what the result of an action will be?
  - Explicit randomness: rolling dice
  - Unpredictable opponents: the ghosts respond randomly
  - Actions can fail: when moving a robot, wheels might slip

- Values should now reflect average-case (expectimax) outcomes, not worst-case (minimax) outcomes

- Expectimax search: compute the average score under optimal play
  - Max nodes as in minimax search
  - Chance nodes are like min nodes but the outcome is uncertain
  - Calculate their expected utilities
  - I.e. take weighted average (expectation) of children

- Later, we’ll learn how to formalize the underlying uncertain-result problems as Markov Decision Processes
Minimax vs Expectimax (Min)

End your misery!
Minimax vs Expectimax (Exp)

Hold on to hope, Pacman!
Expectimax Pseudocode

def value(state):
    if the state is a terminal state: return the state’s utility
    if the next agent is MAX: return max-value(state)
    if the next agent is EXP: return exp-value(state)

def max-value(state):
    initialize $v = -\infty$
    for each successor of state:
        $v = \max(v, \text{value}(\text{successor}))$
    return $v$

def exp-value(state):
    initialize $v = 0$
    for each successor of state:
        $p = \text{probability}(\text{successor})$
        $v += p \times \text{value}(\text{successor})$
    return $v$
Expectimax Pseudocode

def exp-value(state):
    initialize v = 0
    for each successor of state:
        p = probability(successor)
        v += p * value(successor)
    return v

v = (1/2) (8) + (1/3) (24) + (1/6) (-12) = 10
Expectimax Example
Expectimax Pruning?
Depth-Limited Expectimax

Estimate of true expectimax value (which would require a lot of work to compute)
Probabilities
A random variable represents an event whose outcome is unknown
A probability distribution is an assignment of weights to outcomes

Example: Traffic on freeway
- Random variable: T = whether there’s traffic
- Outcomes: T in {none, light, heavy}
- Distribution: \( P(T=\text{none}) = 0.25, P(T=\text{light}) = 0.50, P(T=\text{heavy}) = 0.25 \)

Some laws of probability (more later):
- Probabilities are always non-negative
- Probabilities over all possible outcomes sum to one

As we get more evidence, probabilities may change:
- \( P(T=\text{heavy}) = 0.25, P(T=\text{heavy} \mid \text{Hour=8am}) = 0.60 \)
- We’ll talk about methods for reasoning and updating probabilities later
The expected value of a function of a random variable is the average, weighted by the probability distribution over outcomes.

Example: How long to get to the airport?

<table>
<thead>
<tr>
<th>Time</th>
<th>Probability</th>
<th>Probability Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 min</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>30 min</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>60 min</td>
<td>0.25</td>
<td></td>
</tr>
</tbody>
</table>

\[
\text{Expected Value} = (20 \times 0.25) + (30 \times 0.50) + (60 \times 0.25) = 35 \text{ min}
\]
In expectimax search, we have a probabilistic model of how the opponent (or environment) will behave in any state:

- Model could be a simple uniform distribution (roll a die)
- Model could be sophisticated and require a great deal of computation
- We have a chance node for any outcome out of our control: opponent or environment
- The model might say that adversarial actions are likely!

For now, assume each chance node magically comes along with probabilities that specify the distribution over its outcomes.

Having a probabilistic belief about another agent’s action does not mean that the agent is flipping any coins!
What are Probabilities?

- **Objectivist / frequentist answer:**
  - Averages over repeated experiments
  - E.g. empirically estimating \( P(\text{rain}) \) from historical observation
  - Assertion about how future experiments will go (in the limit)
  - New evidence changes the *reference class*
  - Makes one think of *inherently random* events, like rolling dice

- **Subjectivist / Bayesian answer:**
  - Degrees of belief about unobserved variables
  - E.g. an agent's belief that it's raining, given the temperature
  - E.g. pacman's belief that the ghost will turn left, given the state
  - Often *learn* probabilities from past experiences (more later)
  - New evidence *updates beliefs* (more later)
Quiz: Informed Probabilities

- Let’s say you know that your opponent is actually running a depth 2 minimax, using the result 80% of the time, and moving randomly otherwise.
- Question: What tree search should you use?

Answer: Expectimax!

- To figure out EACH chance node's probabilities, you have to run a simulation of your opponent.
- This kind of thing gets very slow very quickly.
- Even worse if you have to simulate your opponent simulating you...
- ... except for minimax, which has the nice property that it all collapses into one game tree.
The Dangers of Optimism and Pessimism

Dangerous Optimism
Assuming chance when the world is adversarial

Dangerous Pessimism
Assuming the worst case when it’s not likely
Assumptions vs. Reality

Pacman used depth 4 search with an eval function that avoids trouble
Ghost used depth 2 search with an eval function that seeks Pacman
Other Game Types
Mixed Layer Types

- E.g. Backgammon
- Expectiminimax
  - Environment is an extra “random agent” player that moves after each min/max agent
  - Each node computes the appropriate combination of its children
Multi-Agent Utilities

- What if the game is not zero-sum, or has multiple players?

- Generalization of minimax:
  - Terminals have utility tuples
  - Node values are also utility tuples
  - Each player maximizes its own component
  - Can give rise to cooperation and competition dynamically...

```
1,6,6  7,1,2  6,1,2  7,2,1  5,1,7  1,5,2  7,7,1  5,2,5
```
Utilities
Maximum Expected Utility

- Why should we average utilities? Why not minimax?

- Principle of maximum expected utility:
  - A rational agent should choose the action that maximizes its expected utility, given its knowledge

- Questions:
  - Where do utilities come from?
  - How do we know such utilities even exist that represent our preferences?
  - How do we know that averaging even makes sense?
  - What if our behavior (preferences) can’t be described by utilities?
What Utilities to Use?

- For worst-case minimax reasoning, terminal function scale doesn’t matter
  - We just want better states to have higher evaluations (get the ordering right)
  - We call this **insensitivity to monotonic transformations**

- For average-case expectimax reasoning, we need *magnitudes* to be meaningful
Utilities are functions from outcomes (states of the world) to real numbers that describe an agent’s preferences.

Where do utilities come from?
- In a game, may be simple (+1/-1)
- Utilities summarize the agent’s goals
- Theorem: any “rational” preferences can be summarized as a utility function

We hard-wire utilities and let behaviors emerge
- Why don’t we let agents pick utilities?
- Why don’t we prescribe behaviors?
Utilities: Uncertain Outcomes

Getting ice cream

Get Single

Get Double

Oops

Whew!
Preferences

- An agent must have preferences among:
  - Prizes: \( A, B, \) etc.
  - Lotteries: situations with uncertain prizes

\[
L = [p, A; (1 - p), B]
\]

- Notation:
  - Preference:
    \[
    A \succ B
    \]
  - Indifference:
    \[
    A \sim B
    \]
Rationality

PLAN A
- ✔
- ✔
- ✔
- Ø

PLAN B
- ✔
- ✔
- Ø
- Ø
Rational Preferences

- We want some constraints on preferences before we call them rational, such as:

\[
\text{Axiom of Transitivity: } (A \succ B) \land (B \succ C) \Rightarrow (A \succ C)
\]

- For example: an agent with intransitive preferences can be induced to give away all of its money
  - If \( B \succ C \), then an agent with \( C \) would pay (say) 1 cent to get \( B \)
  - If \( A \succ B \), then an agent with \( B \) would pay (say) 1 cent to get \( A \)
  - If \( C \succ A \), then an agent with \( A \) would pay (say) 1 cent to get \( C \)
The Axioms of Rationality

**Orderability**

\[(A \succ B) \lor (B \succ A) \lor (A \sim B)\]

**Transitivity**

\[(A \succ B) \land (B \succ C) \Rightarrow (A \succ C)\]

**Continuity**

\[A \succ B \succ C \Rightarrow \exists p\ [p, A; 1 - p, C] \sim B\]

**Substitutability**

\[A \sim B \Rightarrow [p, A; 1 - p, C] \sim [p, B; 1 - p, C]\]

**Monotonicity**

\[A \succ B \Rightarrow (p \geq q \Leftrightarrow [p, A; 1 - p, B] \geq [q, A; 1 - q, B])\]

Theorem: Rational preferences imply behavior describable as maximization of expected utility
Theorem [Ramsey, 1931; von Neumann & Morgenstern, 1944]
- Given any preferences satisfying these constraints, there exists a real-valued function $U$ such that:

$$U(A) \geq U(B) \iff A \succeq B$$
- i.e. values assigned by $U$ preserve preferences of both prizes and lotteries!

Maximum expected utility (MEU) principle:
- Choose the action that maximizes expected utility
- Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities
- E.g., a lookup table for perfect tic-tac-toe, a reflex vacuum cleaner

Maximum expected utility: $U([p_1, S_1; \ldots ; p_n, S_n]) = \sum_i p_i U(S_i)$
Human Utilities

Spin the wheel or pay $ to pass.
Utility Scales

- Normalized utilities: \( u_+ = 1.0, u_- = 0.0 \)

- **Micromorts**: one-millionth chance of death, useful for paying to reduce product risks, etc.

- **QALYs**: quality-adjusted life years, useful for medical decisions involving substantial risk

- Note: behavior is invariant under positive linear transformation

\[
U'(x) = k_1 U(x) + k_2 \quad \text{where } k_1 > 0
\]

- With deterministic prizes only (no lottery choices), only ordinal utility can be determined, i.e., total order on prizes. To determine magnitudes, must ask questions about lottery preferences.
Utilities map states to real numbers. Which numbers?

Standard approach to assessment (elicitation) of human utilities:

- Compare a prize A to a standard lottery $L_p$ between
  - “best possible prize” $u_+$ with probability $p$
  - “worst possible catastrophe” $u_-$ with probability $1-p$
- Adjust lottery probability $p$ until indifference: $A \sim L_p$
- Resulting $p$ is a utility in $[0,1]$

**Pay $30**

0.999999

0.000001

**No pay**

**Instant death**
Money does not behave as a utility function, but we can talk about the utility of having money (or being in debt).

Given a lottery $L = [p, \text{\$X}; (1-p), \text{\$Y}]$

- The expected monetary value $EMV(L)$ is $p \times \text{\$X} + (1-p) \times \text{\$Y}$
- $U(L) = p \times U(\text{\$X}) + (1-p) \times U(\text{\$Y})$
- Typically, $U(L) < U(EMV(L))$
- In this sense, people are risk-averse
- When deep in debt, people are risk-prone
Example: Insurance

- Consider the lottery \([0.5, \$1000; 0.5, \$0]\)
  - What is its **expected monetary value**? ($500)
  - What is its **certainty equivalent**?
    - Monetary value acceptable in lieu of lottery
    - $400 for most people
  - Difference of $100 is the **insurance premium**
    - There's an insurance industry because people will pay to reduce their risk
    - If everyone were risk-neutral, no insurance needed!
  - It's win-win: you'd rather have the $400 and the insurance company would rather have the lottery (their utility curve is flat and they have many lotteries)
Example: Human Rationality?

- Famous example of Allais (1953)
  - A: [0.8, $4k; 0.2, $0]
  - B: [1.0, $3k; 0.0, $0]
  - C: [0.2, $4k; 0.8, $0]
  - D: [0.25, $3k; 0.75, $0]

- Most people prefer B > A, C > D

- But if $U(0) = 0$, then
  - $B > A$ ⇒ $U(3k) > 0.8 U(4k)$
  - $C > D$ ⇒ $0.8 U(4k) > U(3k)$ (mult both sides by 4 — linear transforms are OK)
Next Time: MDPs!